Cancellation Patterns in Automatic Geometric Theorem Proving

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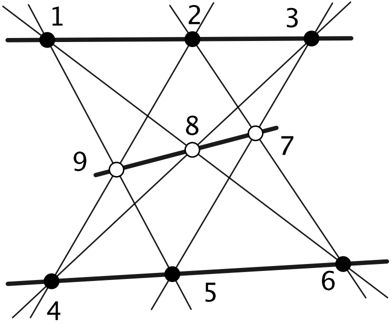
What is this about?

Incidence Theorems in $\mathbb{RP}^2$ like

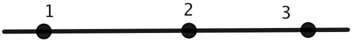


and two algebraic proving techinques which work for *some* theorems.

Both

- make heavily use of cancellation
- are constructed from basic building blocks
- look different at first sight – but surprisingly are the same

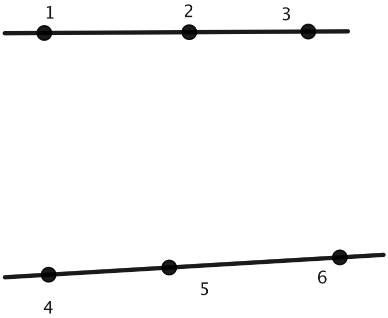
Running example



Collinearity-Hypotheses:

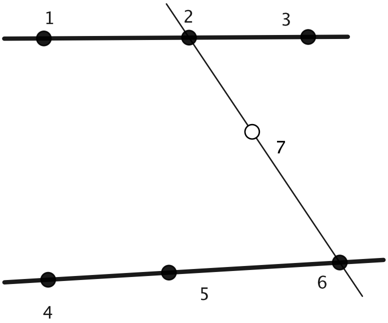
$(1, 2, 3)$, $(4, 5, 6)$,

Running example



Collinearity-Hypotheses:
$(1, 2, 3), (4, 5, 6), (2, 6, 7),$

Running example



Collinearity-Hypotheses:
$(1,2,3),(4,5,6),(2,6,7),(3,5,7),$

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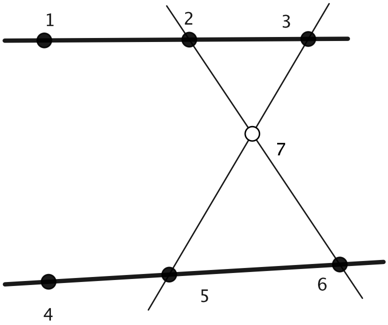
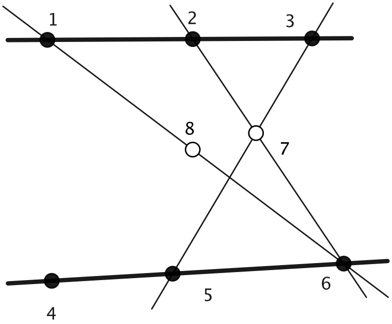


Collinearity-Hypotheses:
$(1,2,3),(4,5,6),(2,6,7),(3,5,7),$
$(1,6,8),$

Running example



Collinearity-Hypotheses:
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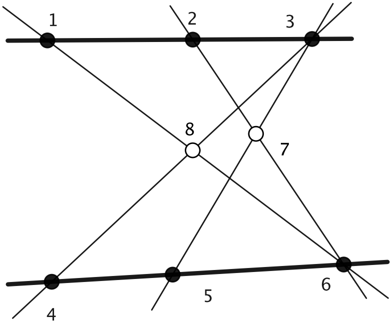
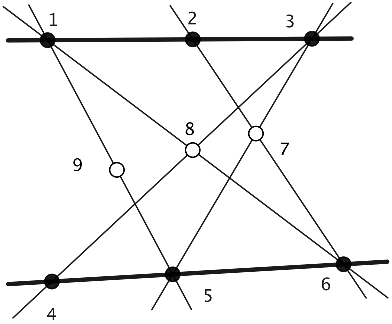


Collinearity-Hypotheses:
$(1,2,3), (4,5,6), (2,6,7), (3,5,7),$
$(1,6,8), (3,4,8),$ $(1,5,9),$

Running example



Collinearity-Hypotheses:

$(1,2,3), (4,5,6), (2,6,7), (3,5,7),$
$(1,6,8), (3,4,8), (1,5,9), (2,4,9)$

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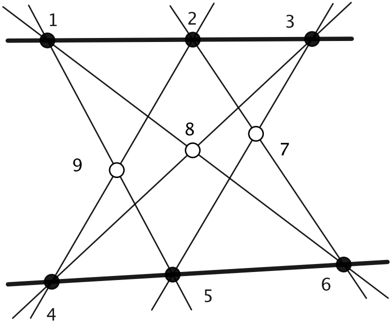

Collinearity-Hypotheses:
$(1,2,3),(4,5,6),(2,6,7),(3,5,7),$
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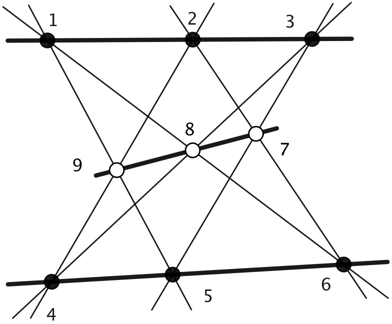
Conclusion:
Point-Triple $(7,8,9)$ is collinear as well!

Running example



Collinearity-Hypotheses:
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later on: non-degeneracy assumptions when needed.

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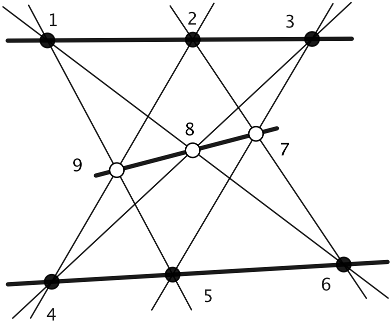

Collinearity-Hypotheses:

$(1, 2, 3), (4, 5, 6), (2, 6, 7), (3, 5, 7), (1, 6, 8), (3, 4, 8), (1, 5, 9), (2, 4, 9)$

Conclusion:

Point-Triple $(7, 8, 9)$ is collinear as well!

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Content

Content

Preliminaries: From Grassmann-Plücker relations to biquadratic equations

Grassmann-Plücker relations: $A, B, C, X, Y \in \mathbb{RP}^2$

$$[ABC][AXY] - [ABX][ACY] + [ABY][ACX] = 0$$

$[* * *]$ – determinant (of homogeneous coordinates) of three points.

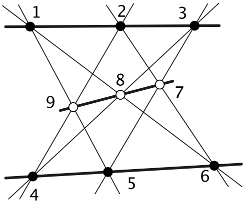
$$\big((A, B, C) \text{ or } (A, X, Y) \quad \text{are collinear}\big) \iff [ABX][ACY] = [ABY][ACX]$$

Definition

$[ABX][ACY] = [ABY][ACX]$ – *biquadratic equations.*

$\rightarrow$ building blocks of binomial proofs.

A binomial Proof for Pappus's Theorem



$$\mathrm{COLL}(123) \quad \Longrightarrow \quad [124][137] = [127][134]$$

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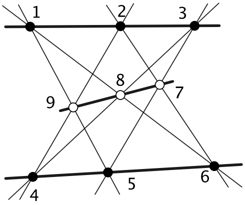

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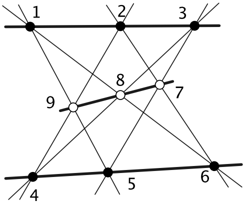

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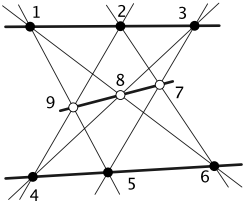

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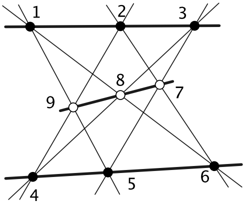

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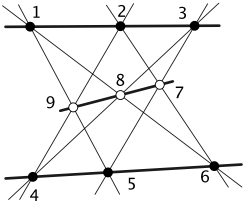

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$(1, 2, 4)$

shall not be collinear

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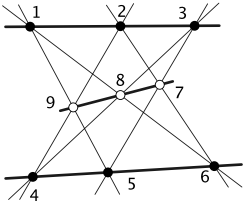

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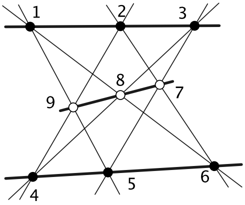

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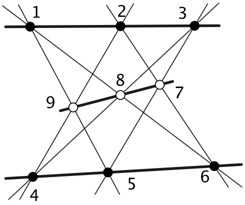

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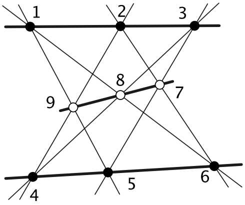

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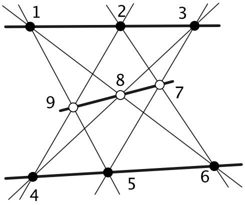

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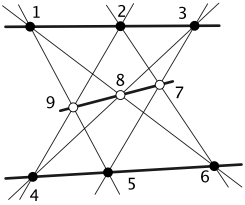

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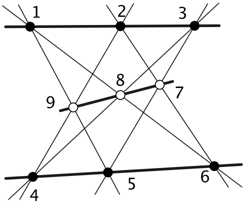

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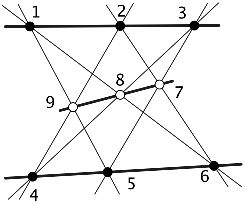

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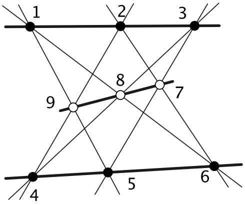

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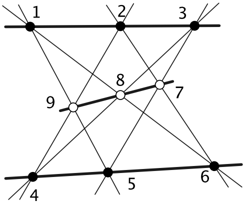

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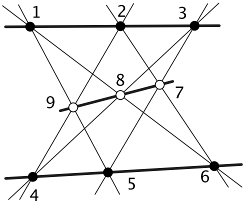

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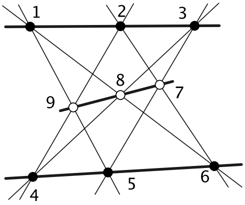

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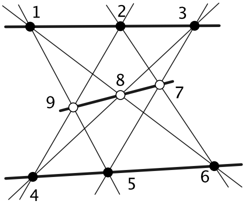
COLL(456)	$\Longrightarrow$	[457][461]	=	[467]
COLL(348)	$\Longrightarrow$	[487]	=	[481][437]
COLL(267)	$\Longrightarrow$	[764]	=	[724][761]
COLL(357)	$\Longrightarrow$	[751][734]	=	[754]
		[791]	=	

We add non-degeneracy assumptions: The triples

$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7)$

shall not be collinear

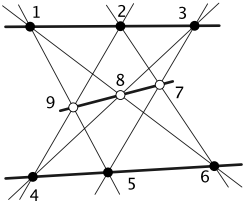
A binomial Proof for Pappus's Theorem



$$
\begin{array}{lcrcl}
\text{COLL}(123) & \Longrightarrow & & = & \\
\text{COLL}(159) & \Longrightarrow & [197] & = & {\color{red}[194]} \\
\text{COLL}(168) & \Longrightarrow & [184][167] & = & [187][164] \\
\text{COLL}(249) & \Longrightarrow & [427]{\color{red}[491]} & = & [497] \\
\text{COLL}(456) & \Longrightarrow & [457][461] & = & [467] \\
\text{COLL}(348) & \Longrightarrow & [487] & = & [481][437] \\
\text{COLL}(267) & \Longrightarrow & [764] & = & [724][761] \\
\text{COLL}(357) & \Longrightarrow & [734] & = & [754] \\
\hline
 & - & [791] & = &
\end{array}
$$

We add non-degeneracy assumptions: The triples
$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4)$
shall not be collinear

A binomial Proof for Pappus's Theorem



$$
\begin{array}{lcrcl}
\mathrm{COLL}(123) & \Longrightarrow & & = & \\
\mathrm{COLL}(159) & \Longrightarrow & [197] & = & \\
\mathrm{COLL}(168) & \Longrightarrow & {\color{red}[184]}[167] & = & [187][164] \\
\mathrm{COLL}(249) & \Longrightarrow & [427] & = & [497] \\
\mathrm{COLL}(456) & \Longrightarrow & [457][461] & = & [467] \\
\mathrm{COLL}(348) & \Longrightarrow & [487] & = & {\color{red}[481]}[437] \\
\mathrm{COLL}(267) & \Longrightarrow & [764] & = & [724][761] \\
\mathrm{COLL}(357) & \Longrightarrow & [734] & = & [754] \\
\hline
 & - & [791] & = &
\end{array}
$$

We add non-degeneracy assumptions: The triples

$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4)$

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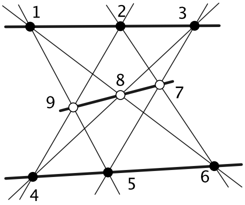
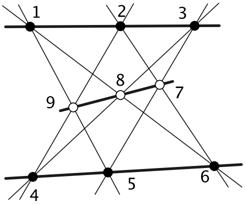
A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$		=	
COLL(159)	$\Longrightarrow$	[197]	=	
COLL(168)	$\Longrightarrow$	[167]	=	[187][164]
COLL(249)	$\Longrightarrow$	[427]	=	[497]
COLL(456)	$\Longrightarrow$	[457][461]	=	[467]
COLL(348)	$\Longrightarrow$	[487]	=	[437]
COLL(267)	$\Longrightarrow$	[764]	=	[724][761]
COLL(357)	$\Longrightarrow$	[734]	=	[754]
	–	[791]	=	

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4)$, $(1,6,7)$ shall not be collinear

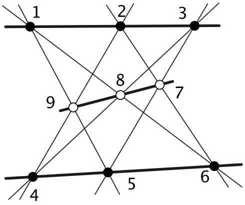
A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$		=	
COLL(159)	$\Longrightarrow$	[197]	=	
COLL(168)	$\Longrightarrow$		=	[187][164]
COLL(249)	$\Longrightarrow$	[427]	=	[497]
COLL(456)	$\Longrightarrow$	[457][461]	=	[467]
COLL(348)	$\Longrightarrow$	[487]	=	[437]
COLL(267)	$\Longrightarrow$	[764]	=	[724]
COLL(357)	$\Longrightarrow$	[734]	=	[754]
		[791]	=	

We add non-degeneracy assumptions: The triples
$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$
$(1,6,7)$ shall not be collinear

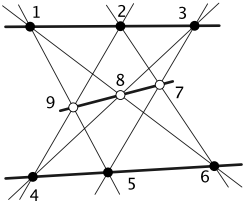
A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$		=	
COLL(159)	$\Longrightarrow$	[197]	=	
COLL(168)	$\Longrightarrow$		=	[187][164]
COLL(249)	$\Longrightarrow$	[427]	=	[497]
COLL(456)	$\Longrightarrow$	[457][461]	=	[467]
COLL(348)	$\Longrightarrow$	[487]	=	[437]
COLL(267)	$\Longrightarrow$	[764]	=	[724]
COLL(357)	$\Longrightarrow$	[734]	=	[754]
	−	[791]	=	[781]

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$ $(1,6,7)$, $(1,6,4),\ldots$ shall not be collinear

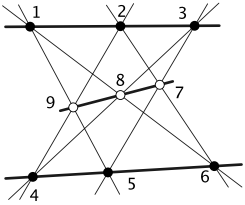
A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$			=	
COLL(159)	$\Longrightarrow$		[197]	=	
COLL(168)	$\Longrightarrow$			=	[187]
COLL(249)	$\Longrightarrow$	[427]		=	[497]
COLL(456)	$\Longrightarrow$	[457]		=	[467]
COLL(348)	$\Longrightarrow$	[487]		=	[437]
COLL(267)	$\Longrightarrow$		[764]	=	[724]
COLL(357)	$\Longrightarrow$		[734]	=	[754]
			[791]	=	[781]

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$ $(1,6,7),(1,6,4),\ldots$ shall not be collinear

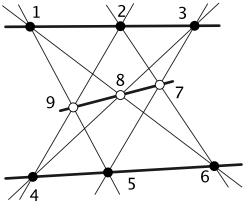
A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$			=		
COLL(159)	$\Longrightarrow$		[197]	=		
COLL(168)	$\Longrightarrow$			=	[187]	
COLL(249)	$\Longrightarrow$			=		[497]
COLL(456)	$\Longrightarrow$	[457]		=		[467]
COLL(348)	$\Longrightarrow$	[487]		=		[437]
COLL(267)	$\Longrightarrow$		[764]	=		
COLL(357)	$\Longrightarrow$		[734]	=	[754]	
		–	[791]	=	[781]	

We add non-degeneracy assumptions: The triples
$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$
$(1,6,7),(1,6,4),\ldots$ shall not be collinear

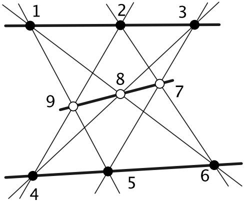
A binomial Proof for Pappus's Theorem



$$
\begin{array}{lcllcll}
\mathrm{COLL}(123) & \Longrightarrow & & & = & & \\
\mathrm{COLL}(159) & \Longrightarrow & & [197] & = & & \\
\mathrm{COLL}(168) & \Longrightarrow & & & = & [187] & \\
\mathrm{COLL}(249) & \Longrightarrow & & & = & & [497] \\
\mathrm{COLL}(456) & \Longrightarrow & [457] & & = & & [467] \\
\mathrm{COLL}(348) & \Longrightarrow & [487] & & = & & [437] \\
\mathrm{COLL}(267) & \Longrightarrow & & [764] & = & & \\
\mathrm{COLL}(357) & \Longrightarrow & & [734] & = & [754] & \\
\hline
 & & & [791] & = & [781][794] &
\end{array}
$$

We add non-degeneracy assumptions: The triples
$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$
$(1,6,7),(1,6,4),\ldots$ shall not be collinear

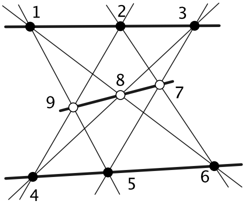
A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$				=		
COLL(159)	$\Longrightarrow$			[197]	=		
COLL(168)	$\Longrightarrow$				=	[187]	
COLL(249)	$\Longrightarrow$				=		[497]
COLL(456)	$\Longrightarrow$				=		[467]
COLL(348)	$\Longrightarrow$		[487]		=		[437]
COLL(267)	$\Longrightarrow$			[764]	=		
COLL(357)	$\Longrightarrow$			[734]	=		
		–		[791]	=	[781]	[794]

We add non-degeneracy assumptions: The triples
$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$
$(1,6,7),(1,6,4),\ldots$ shall not be collinear

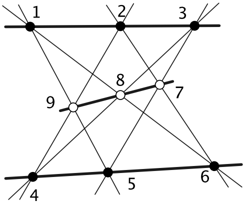
A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$			=		
COLL(159)	$\Longrightarrow$		[197]	=		
COLL(168)	$\Longrightarrow$			=	[187]	
COLL(249)	$\Longrightarrow$			=		[497]
COLL(456)	$\Longrightarrow$			=		
COLL(348)	$\Longrightarrow$	[487]		=		[437]
COLL(267)	$\Longrightarrow$			=		
COLL(357)	$\Longrightarrow$		[734]	=		
			[791]	=	[781][794]	

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$ $(1,6,7),(1,6,4),\ldots$ shall not be collinear

A binomial Proof for Pappus's Theorem

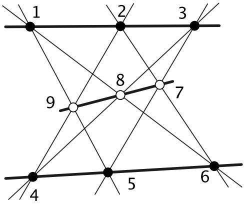


COLL(123)	$\Longrightarrow$			=		
COLL(159)	$\Longrightarrow$		[197]	=		
COLL(168)	$\Longrightarrow$			=	[187]	
COLL(249)	$\Longrightarrow$			=		[497]
COLL(456)	$\Longrightarrow$			=		
COLL(348)	$\Longrightarrow$	[487]		=		[437]
COLL(267)	$\Longrightarrow$			=		
COLL(357)	$\Longrightarrow$		[734]	=		

$$-[784][791] = [781][794]$$

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),(1,6,7),(1,6,4),\ldots$ shall not be collinear

A binomial Proof for Pappus's Theorem

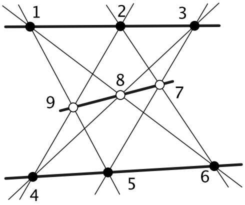


COLL(123)	$\Longrightarrow$			=	
COLL(159)	$\Longrightarrow$		[197]	=	
COLL(168)	$\Longrightarrow$			=	[187]
COLL(249)	$\Longrightarrow$			=	[497]
COLL(456)	$\Longrightarrow$			=	
COLL(348)	$\Longrightarrow$	[487]		=	
COLL(267)	$\Longrightarrow$			=	
COLL(357)	$\Longrightarrow$			=	

[784][791] = [781][794]

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$ $(1,6,7),(1,6,4),\dots$ shall not be collinear

A binomial Proof for Pappus's Theorem

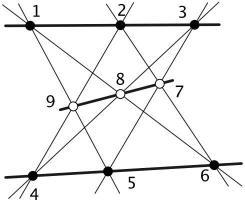


COLL(123)	$\Longrightarrow$			=		
COLL(159)	$\Longrightarrow$		[197]	=		
COLL(168)	$\Longrightarrow$			=	[187]	
COLL(249)	$\Longrightarrow$			=		[497]
COLL(456)	$\Longrightarrow$			=		
COLL(348)	$\Longrightarrow$	[487]		=		
COLL(267)	$\Longrightarrow$			=		
COLL(357)	$\Longrightarrow$			=		

COLL(789) or COLL(714) $\Longleftarrow$ $-\underbrace{[784][791] = [781][794]}_{\text{biqu. equ.}}$

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),(1,6,7),(1,6,4),\ldots$ shall not be collinear

A binomial Proof for Pappus's Theorem

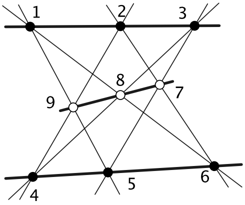


COLL(123)	$\Longrightarrow$			=		
COLL(159)	$\Longrightarrow$		[197]	=		
COLL(168)	$\Longrightarrow$			=	[187]	
COLL(249)	$\Longrightarrow$			=		[497]
COLL(456)	$\Longrightarrow$			=		
COLL(348)	$\Longrightarrow$	[487]		=		
COLL(267)	$\Longrightarrow$			=		
COLL(357)	$\Longrightarrow$			=		

COLL(789) or COLL(714) $\Longleftarrow$ $\underbrace{[784][791] = [781][794]}_{\text{biqu. equ.}}$

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$ $(1,6,7),(1,6,4),\ldots,(7,1,4)$ shall not be collinear

A binomial Proof for Pappus's Theorem



COLL(123)	$\Longrightarrow$			=	
COLL(159)	$\Longrightarrow$		[197]	=	
COLL(168)	$\Longrightarrow$			=	[187]
COLL(249)	$\Longrightarrow$			=	[497]
COLL(456)	$\Longrightarrow$			=	
COLL(348)	$\Longrightarrow$	[487]		=	
COLL(267)	$\Longrightarrow$			=	
COLL(357)	$\Longrightarrow$			=	

COLL(789) or COLL(714) $\Longleftarrow$ $\underbrace{[784][791] = [781][794]}_{\text{biqu. equ.}}$

We add non-degeneracy assumptions: The triples $(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),(1,6,7),(1,6,4),\dots,(7,1,4)$ shall not be collinear

Natural representation of incidence theorems

Incidence Theorems

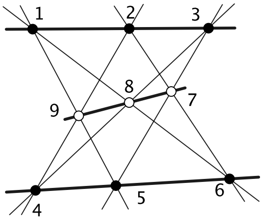
$\mathcal{T} = (\mathbf{H}, \mathbf{B}, C)$.

$\mathbf{H}$ – collinearity hypotheses
$\mathbf{B}$ – non-degeneracy assumptions (bases)
C – Conclusion

encoded by triples (of indices) of points.

How to identify the non-degeneracy assumptions **B**?

Recall:



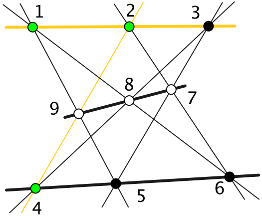
$$
\begin{array}{lcl}
\text{COLL}(123) & \Longrightarrow & [124][137] = [127][134] \\
\text{COLL}(159) & \Longrightarrow & [154][197] = [157][194] \\
\text{COLL}(168) & \Longrightarrow & [184][167] = [187][164] \\
\text{COLL}(249) & \Longrightarrow & [427][491] = [421][497] \\
\text{COLL}(456) & \Longrightarrow & [457][461] = [451][467] \\
\text{COLL}(348) & \Longrightarrow & [487][431] = [481][437] \\
\text{COLL}(267) & \Longrightarrow & [721][764] = [724][761] \\
\text{COLL}(357) & \Longrightarrow & [751][734] = [754][731] \\
\hline
\text{COLL}(789) \text{ or} & \Longleftarrow & [784][791] = [781][794] \\
\text{COLL}(714) & &
\end{array}
$$

non-degeneracy triples:

$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$
$(1,6,7),(1,6,4),\ldots$ and $(7,1,4)$

How to identify the non-degeneracy assumptions **B**?

Recall:



$$
\begin{array}{lcl}
\mathrm{COLL}(123) & \Longrightarrow & [124][137] = [127][134] \\
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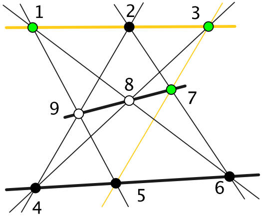
\hline
\mathrm{COLL}(789) \text{ or } \mathrm{COLL}(714) & \Longleftarrow & [784][791] = [781][794]
\end{array}
$$

non-degeneracy triples:

$(1,2,4), (1,3,7), (1,2,7), (1,3,4), (1,5,4), (1,5,7), (1,9,4), (1,8,4), (1,6,7), (1,6,4), \ldots$ and $(7,1,4)$

How to identify the non-degeneracy assumptions **B**?

Recall:



COLL(123)	$\Longrightarrow$	[124][137] = [127][134]
COLL(159)	$\Longrightarrow$	[154][197] = [157][194]
COLL(168)	$\Longrightarrow$	[184][167] = [187][164]
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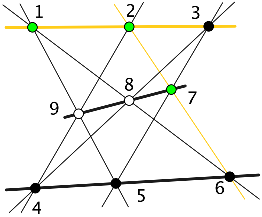
COLL(348)	$\Longrightarrow$	[487][431] = [481][437]
COLL(267)	$\Longrightarrow$	[721][764] = [724][761]
COLL(357)	$\Longrightarrow$	[751][734] = [754][731]
COLL(789) or COLL(714)	$\Longleftarrow$	[784][791] = [781][794]

non-degeneracy triples:

$(1,2,4), (1,3,7), (1,2,7), (1,3,4), (1,5,4), (1,5,7), (1,9,4), (1,8,4),$
$(1,6,7), (1,6,4), \ldots$ and $(7,1,4)$

How to identify the non-degeneracy assumptions **B**?

Recall:



$$\begin{array}{lcl}
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\mathrm{COLL}(159) & \Longrightarrow & [154][197] = [157][194] \\
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\mathrm{COLL}(249) & \Longrightarrow & [427][491] = [421][497] \\
\mathrm{COLL}(456) & \Longrightarrow & [457][461] = [451][467] \\
\mathrm{COLL}(348) & \Longrightarrow & [487][431] = [481][437] \\
\mathrm{COLL}(267) & \Longrightarrow & [721][764] = [724][761] \\
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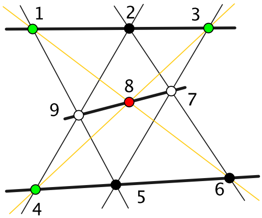
\hline
\mathrm{COLL}(789) \text{ or } \mathrm{COLL}(714) & \Longleftarrow & [784][791] = [781][794]
\end{array}$$

non-degeneracy triples:

$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$
$(1,6,7),(1,6,4),\dots$ and $(7,1,4)$

How to identify the non-degeneracy assumptions **B**?

Recall:



$$\begin{array}{lcl}
\mathrm{COLL}(123) & \Longrightarrow & [124][137]=[127][134] \\
\mathrm{COLL}(159) & \Longrightarrow & [154][197]=[157][194] \\
\mathrm{COLL}(168) & \Longrightarrow & [184][167]=[187][164] \\
\mathrm{COLL}(249) & \Longrightarrow & [427][491]=[421][497] \\
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\mathrm{COLL}(267) & \Longrightarrow & [721][764]=[724][761] \\
\mathrm{COLL}(357) & \Longrightarrow & [751][734]=[754][731] \\
\hline
\mathrm{COLL}(789) \text{ or} & \Longleftarrow & [784][791]=[781][794] \\
\mathrm{COLL}(714) & &
\end{array}$$

non-degeneracy triples:

$(1,2,4),(1,3,7),(1,2,7),(1,3,4),(1,5,4),(1,5,7),(1,9,4),(1,8,4),$
$(1,6,7),(1,6,4),\ldots$ and $(7,1,4)$

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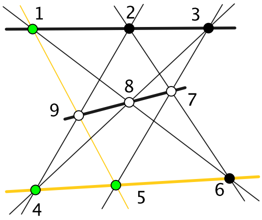

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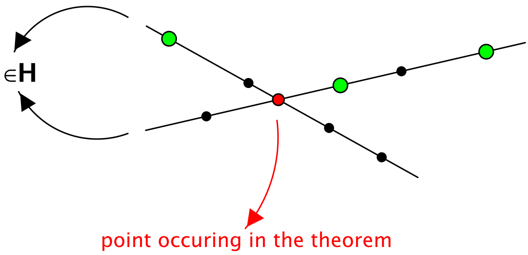
How to identify the non-degeneracy assumptions **B**?

Triples of points in **B**:

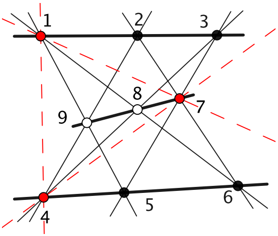


So:

To each incidence theorem, there are reasonable intrinsic and algorithmically good-natured non-degeneracy assumptions!

How to identify the non-degeneracy assumptions **B**?

Recall:



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non-degeneracy triples:

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$(1,6,7),(1,6,4),\ldots$ and $(7,1,4)$

How to find a proof

- state *all* biquadratic equations, e.g. $[ABX][ACY] = [ABY][ACX]$
- take the logarithm $\rightsquigarrow \log[ABX] + \log[ACY] = \log[ABY] + \log[ACX]$
- $\log[***]$'s $\rightarrow$ formal symbols
- solve a linear equation system

$\rightarrow$ polynomial algorithm (R-G)

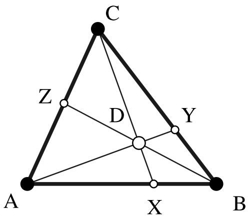
Subtility: permutations inside the brackets

Content

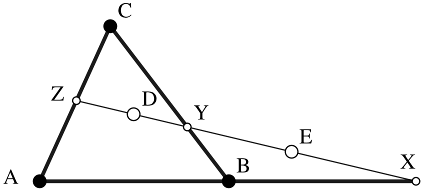
Building blocks for this Proof

In an affine Setup with oriented lengths:

Ceva's theorem



Menelaus's theorem



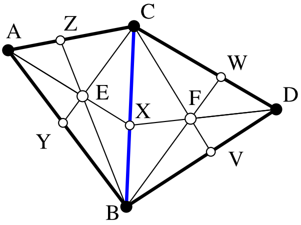
$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1$$

$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = -1$$

one edge of the triangle $\leftrightarrow$ one ratio in the formula

$\longrightarrow$ building blocks for Ceva-Menelaus proofs!

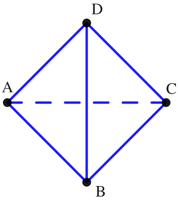
Affine Cancellation



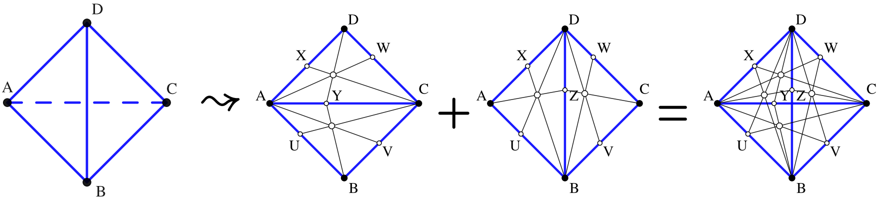
$$\frac{|AY|}{|YB|}\cdot\frac{|BX|}{|XC|}\cdot\frac{|CZ|}{|ZA|}\cdot\frac{|BV|}{|VD|}\cdot\frac{|DW|}{|WC|}\cdot\frac{|CX|}{|XB|}=1 \quad\Leftrightarrow\quad \frac{|AY|}{|YB|}\cdot\frac{|CZ|}{|ZA|}\cdot\frac{|BV|}{|VD|}\cdot\frac{|DW|}{|WC|}=1$$

Ceva-Menelaus Proofs

Ceva-Menelaus Proofs



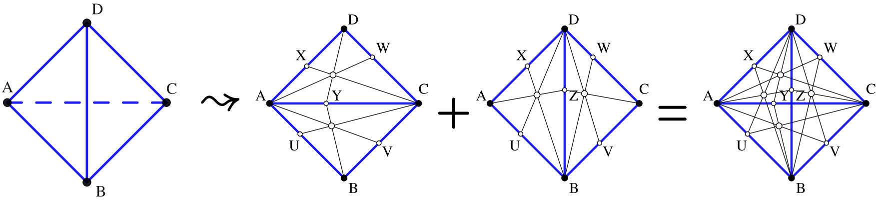
Ceva-Menelaus Proofs



$$\frac{|AU|}{|UB|}\cdot\frac{|BV|}{|VC|}\cdot\frac{|CY|}{|YA|}\quad\cdot\quad\frac{|CW|}{|WD|}\cdot\frac{|DX|}{|XA|}\cdot\frac{|AY|}{|YC|}\quad\cdot$$

$$\frac{|AX|}{|XD|}\cdot\frac{|DZ|}{|ZB|}\cdot\frac{|BU|}{|UA|}\quad\cdot\quad\frac{|BZ|}{|ZD|}\cdot\frac{|DW|}{|WC|}\cdot\frac{|CV|}{|VB|}=1.$$

Ceva-Menelaus Proofs



$$\frac{|AU|}{|UB|} \cdot \frac{|BV|}{|VC|} \cdot \frac{|CY|}{|YA|} \quad \cdot \quad \frac{|CW|}{|WD|} \cdot \frac{|DX|}{|XA|} \cdot \frac{|AY|}{|YC|} \quad \cdot$$

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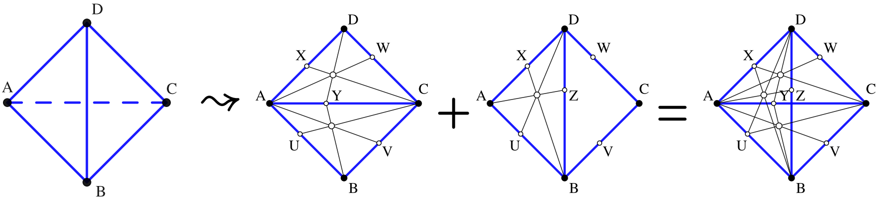
Ceva-Menelaus Proofs



$$\underbrace{\frac{|AU|}{|UB|}\cdot\frac{|BV|}{|VC|}\cdot\frac{|CY|}{|YA|}}_{=-1} \quad\cdot\quad \underbrace{\frac{|CW|}{|WD|}\cdot\frac{|DX|}{|XA|}\cdot\frac{|AY|}{|YC|}}_{=-1} \quad\cdot$$

$$\underbrace{\frac{|AX|}{|XD|}\cdot\frac{|DZ|}{|ZB|}\cdot\frac{|BU|}{|UA|}}_{=-1} \quad\cdot\quad \frac{|BZ|}{|ZD|}\cdot\frac{|DW|}{|WC|}\cdot\frac{|CV|}{|VB|} \quad = 1.$$

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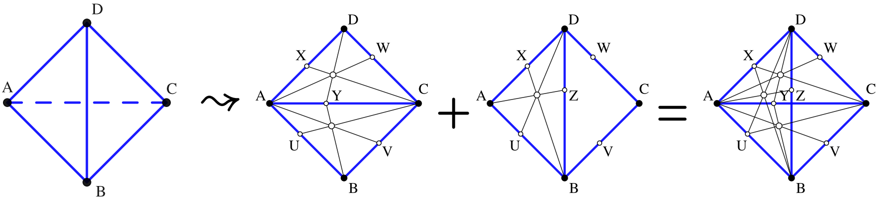

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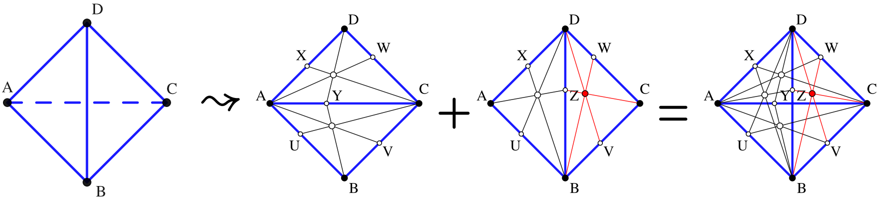
Ceva-Menelaus Proofs



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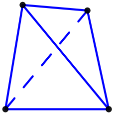

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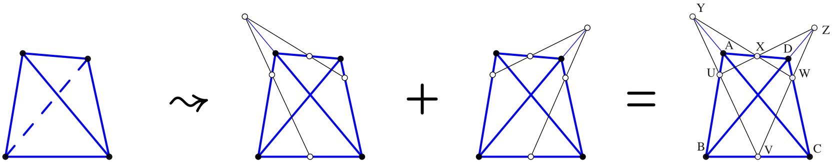
Ceva-Menelaus Proofs



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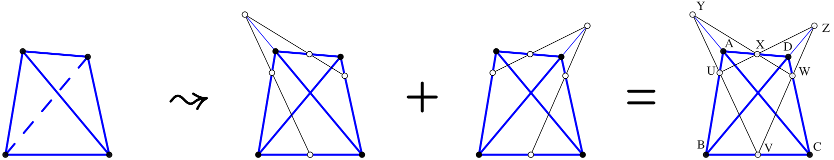

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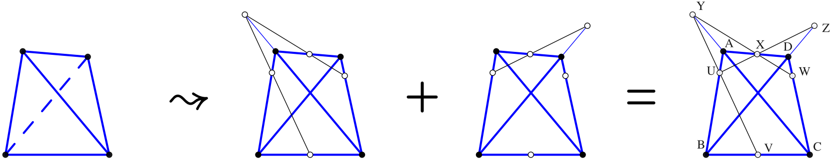
Ceva-Menelaus Proofs



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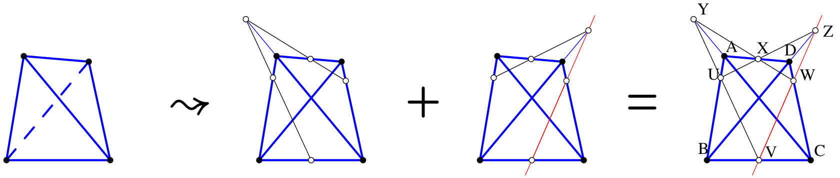
Ceva-Menelaus Proofs



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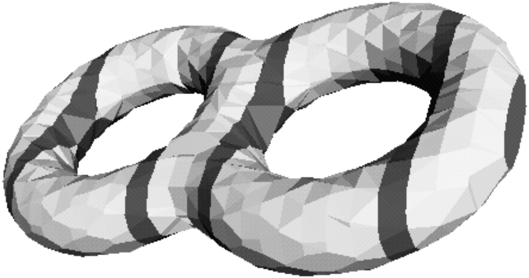
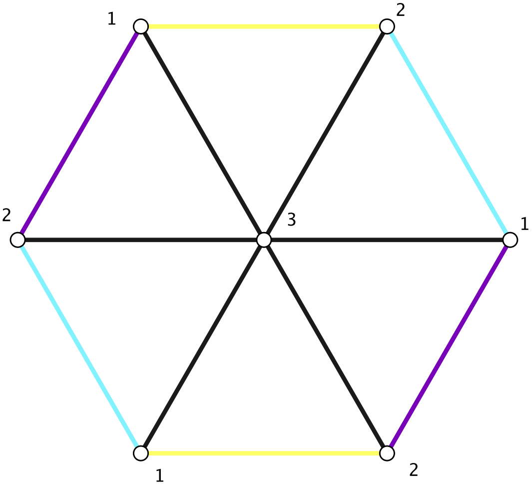
Ceva-Menelaus Proofs



Ceva-Menelaus Proofs

A “small” triangulation of a torus:



Content

Reconsider the binomial proof of Pappus's Theorem

$$[124][137] = [127][134] \iff \frac{[124][137]}{[127][134]} = 1$$

$$[154][197] = [157][194] \iff \frac{[154][197]}{[157][194]} = 1$$

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$$[487][431] = [481][437] \iff \frac{[487][431]}{[481][437]} = 1$$

$$[721][764] = [724][761] \iff \frac{[721][764]}{[724][761]} = 1$$

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$$\frac{[124]}{[127]} \cdot \frac{[137]}{[134]} \cdot \frac{[154]}{[157]} \cdot \frac{[197]}{[194]} \cdot \frac{[184]}{[187]} \cdot \frac{[167]}{[164]} \cdot \frac{[427]}{[421]} \cdot \frac{[491]}{[497]} \cdot \frac{[457]}{[451]} \cdot \frac{[461]}{[467]} \cdot \frac{[487]}{[481]} \cdot \frac{[431]}{[437]} \cdot \frac{[721]}{[724]} \cdot \frac{[764]}{[761]} \cdot \frac{[751]}{[754]} \cdot \frac{[734]}{[731]} \cdot \frac{[781]}{[784]} \cdot \frac{[794]}{[791]} = 1$$

Deriving a graph

$$\frac{[124]}{[127]} \cdot \frac{[137]}{[134]} \cdot \frac{[154]}{[157]} \cdot \frac{[197]}{[194]} \cdot \frac{[184]}{[187]} \cdot \frac{[167]}{[164]} \cdot \frac{[427]}{[421]} \cdot \frac{[491]}{[497]} \cdot \frac{[457]}{[451]} \cdot \frac{[461]}{[467]} \cdot \frac{[487]}{[481]} \cdot \frac{[431]}{[437]} \cdot \frac{[721]}{[724]} \cdot \frac{[764]}{[761]} \cdot \frac{[751]}{[754]} \cdot \frac{[734]}{[731]} \cdot \frac{[781]}{[784]} \cdot \frac{[794]}{[791]} = 1$$

$\Updownarrow$

Deriving a graph

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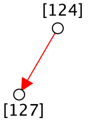

Deriving a graph

$$\frac{[124]}{[127]}\cdot\frac{[137]}{[134]}\cdot\frac{[154]}{[157]}\cdot\frac{[197]}{[194]}\cdot\frac{[184]}{[187]}\cdot\frac{[167]}{[164]}\cdot\frac{[427]}{[421]}\cdot\frac{[491]}{[497]}\cdot\frac{[457]}{[451]}\cdot\frac{[461]}{[467]}\cdot$$
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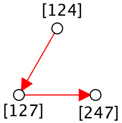

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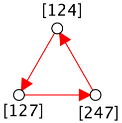

→ Graph: vertices: (non-zero) brackets
edges: between vertices differing in one index

Deriving a graph

$$\frac{[124]}{[127]}\cdot\frac{[137]}{[134]}\cdot\frac{[154]}{[157]}\cdot\frac{[197]}{[194]}\cdot\frac{[184]}{[187]}\cdot\frac{[167]}{[164]}\cdot\frac{[427]}{[421]}\cdot\frac{[491]}{[497]}\cdot\frac{[457]}{[451]}\cdot\frac{[461]}{[467]}\cdot$$
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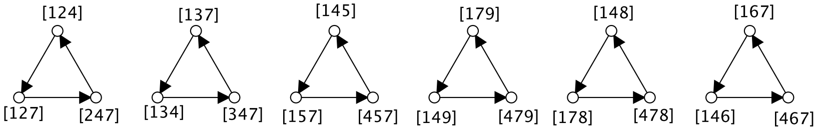
$\leftrightsquigarrow$



→ Graph:
vertices: (non-zero) brackets
edges: between vertices differing in one index

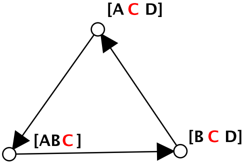
Roadmap

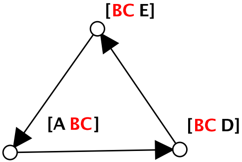
Two tasks:

- Associating graph triangles with Ceva and Menelaus triangles
- Finding triangles in every such graph!

Interpreting graph-triangles

Two combinatorial kinds of triangles:





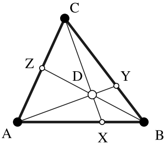
Graph-triangles will be translated to Ceva and Menelaus triangles:

Ceva!

Menelaus!

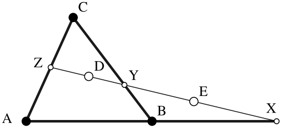
From lengths to brackets

Ceva's theorem



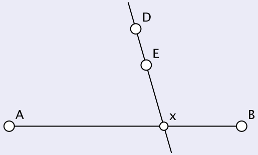
$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = 1$$

Menelaus's theorem



$$\frac{|AX|}{|XB|} \cdot \frac{|BY|}{|YC|} \cdot \frac{|CZ|}{|ZA|} = -1$$

A Tool:



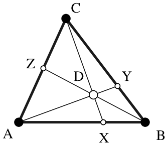
affine $\frac{|Ax|}{|xB|} = \frac{[A\ DE]}{[B\ ED]}$ (projective)

$$\frac{[A\ DC]}{[B\ CD]} \cdot \frac{[B\ AD]}{[C\ DA]} \cdot \frac{[C\ DB]}{[A\ BD]} = 1$$

$$\frac{[A\ DE]}{[B\ ED]} \cdot \frac{[B\ DE]}{[C\ ED]} \cdot \frac{[C\ DE]}{[A\ ED]} = -1$$

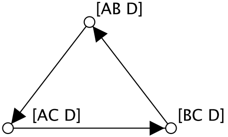
The base triangles corresponding to Ceva and Menelaus triangles

Ceva's theorem

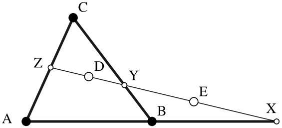


$$\frac{[A\ \ DC]}{[B\ \ CD]} \cdot \frac{[B\ \ AD]}{[C\ \ DA]} \cdot \frac{[C\ \ DB]}{[A\ \ BD]} = 1$$

$$\Downarrow$$

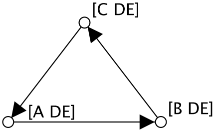


Menelaus' theorem

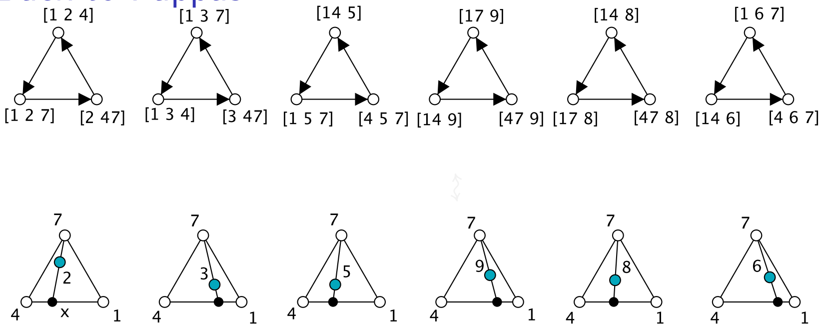


$$\frac{[A\ \ DE]}{[B\ \ ED]} \cdot \frac{[B\ \ DE]}{[C\ \ ED]} \cdot \frac{[C\ \ DE]}{[A\ \ ED]} = -1$$

$$\Downarrow$$



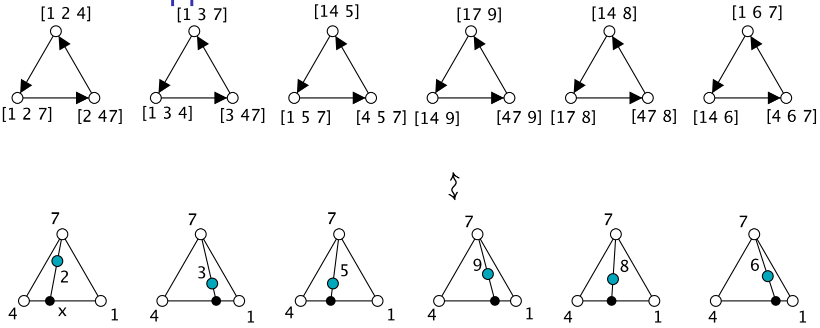
Back to Pappus



Which triangles should be glued together? – We know:

$$[721][764] = [724][761] \quad \Leftrightarrow \quad \frac{[721][764]}{[274][671]} = 1 \quad \Leftrightarrow \quad \frac{[1\ 72]}{[4\ 27]} = \frac{[1\ 67]}{[4\ 76]}$$

Back to Pappus



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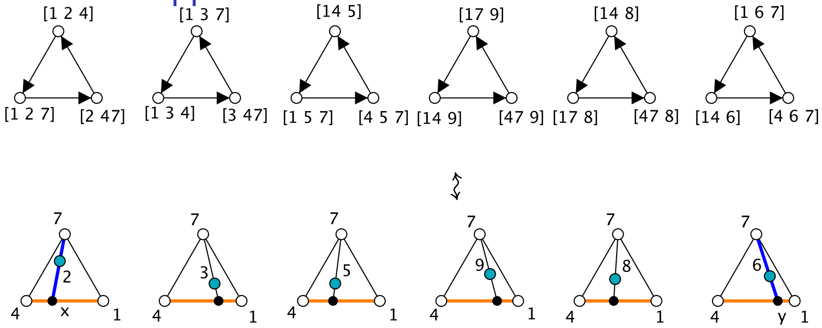
Back to Pappus



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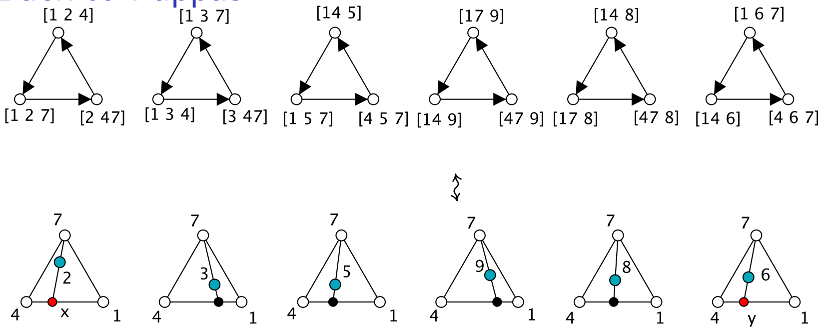
$$\Leftrightarrow \quad \frac{|1x|}{|x4|} = \frac{|1y|}{|y4|} \quad \Leftrightarrow \quad x = y$$

Back to Pappus



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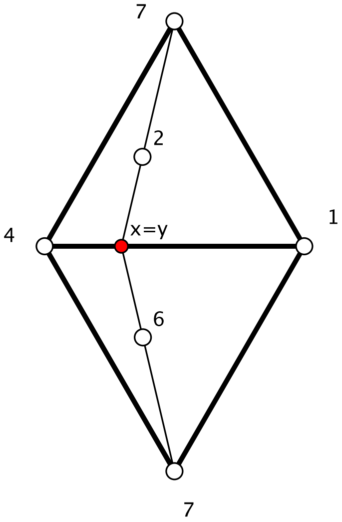
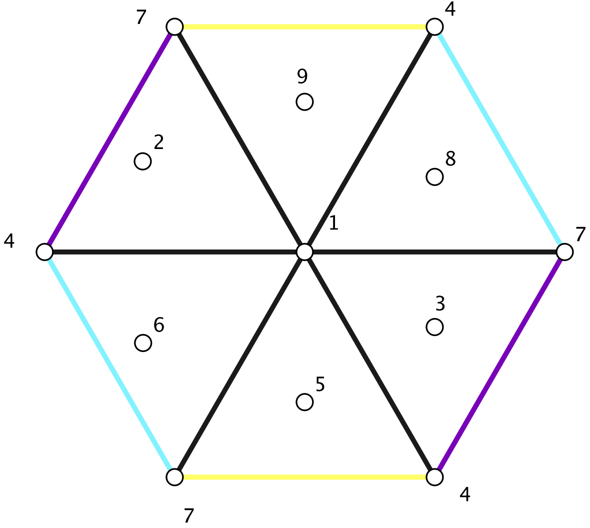
Back to Pappus

So:



Back to Pappus



Roadmap

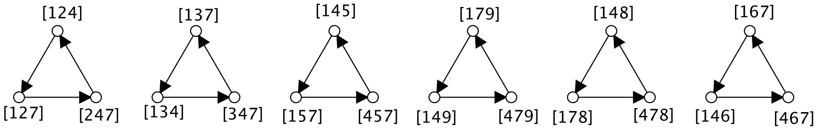
Two tasks:

- Associating graph triangles with Ceva and Menelaus triangles
- Finding triangles in every such graph!

On finding triangles in every such a graph

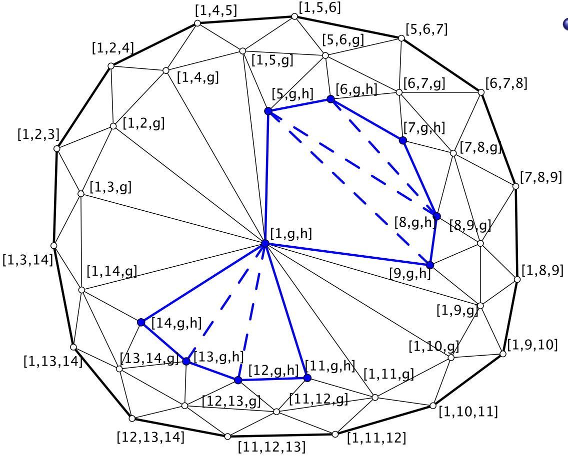
Pappus: misleading example!

misleading picture:



In general

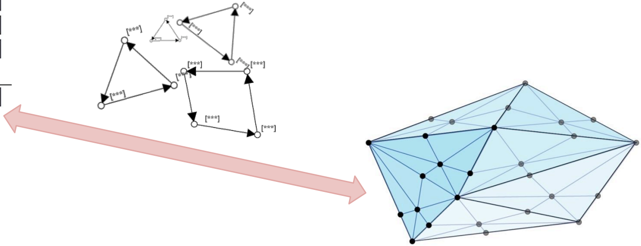
- n-cycles
- 2 generic points $g, h \rightsquigarrow$ new vertices in the graph



- It can be shown: number of Menelaus triangles is even

Conclusion

$$\begin{array}{c} [* * *][* * *] = [* * *][* * *] \\ [* * *][* * *] = [* * *][* * *] \\ [* * *][* * *] = [* * *][* * *] \\ \vdots \\ \hline [* * *][* * *] = [* * *][* * *] \end{array}$$



Binomial Proofs	Ceva-Menelaus Proofs
cancellation patterns on $[* * *]$	cancellation patterns on edges
easy to verify	visual proofs
algorithmically feasible	topological information
algebraic flavor	geometric flavor