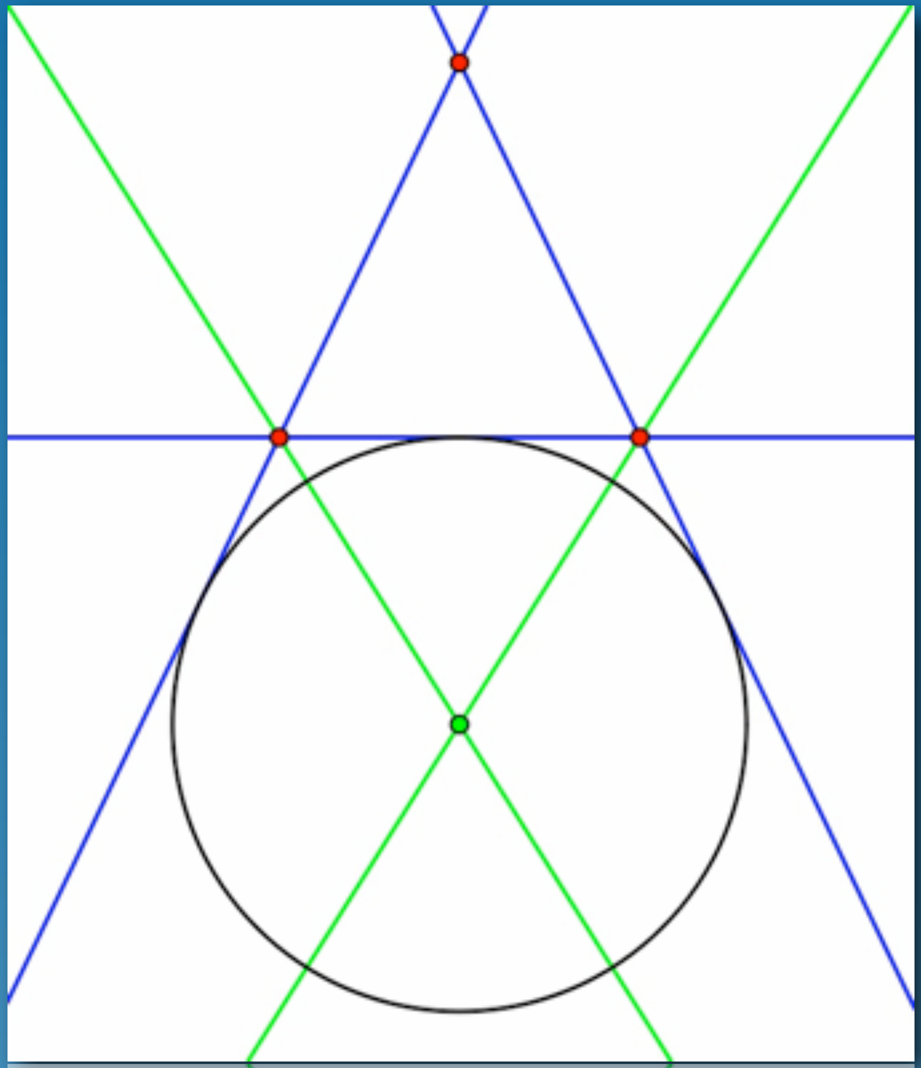
On the Complexity of the Reachability Problem in Dynamic Geometry

Thorsten Orendt

The Reachability Problem (*Reach*)

Can you walk continuously from one instance to another specific one?



starting instance

terminal instance

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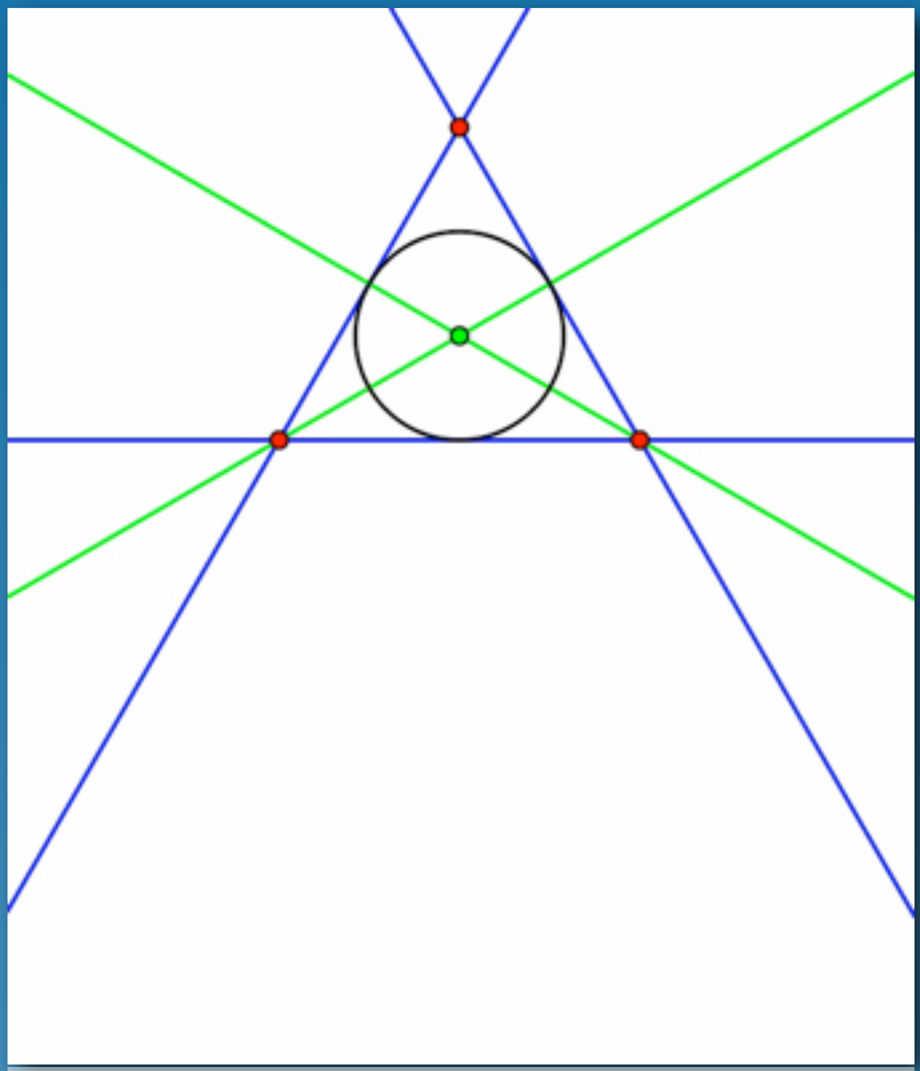

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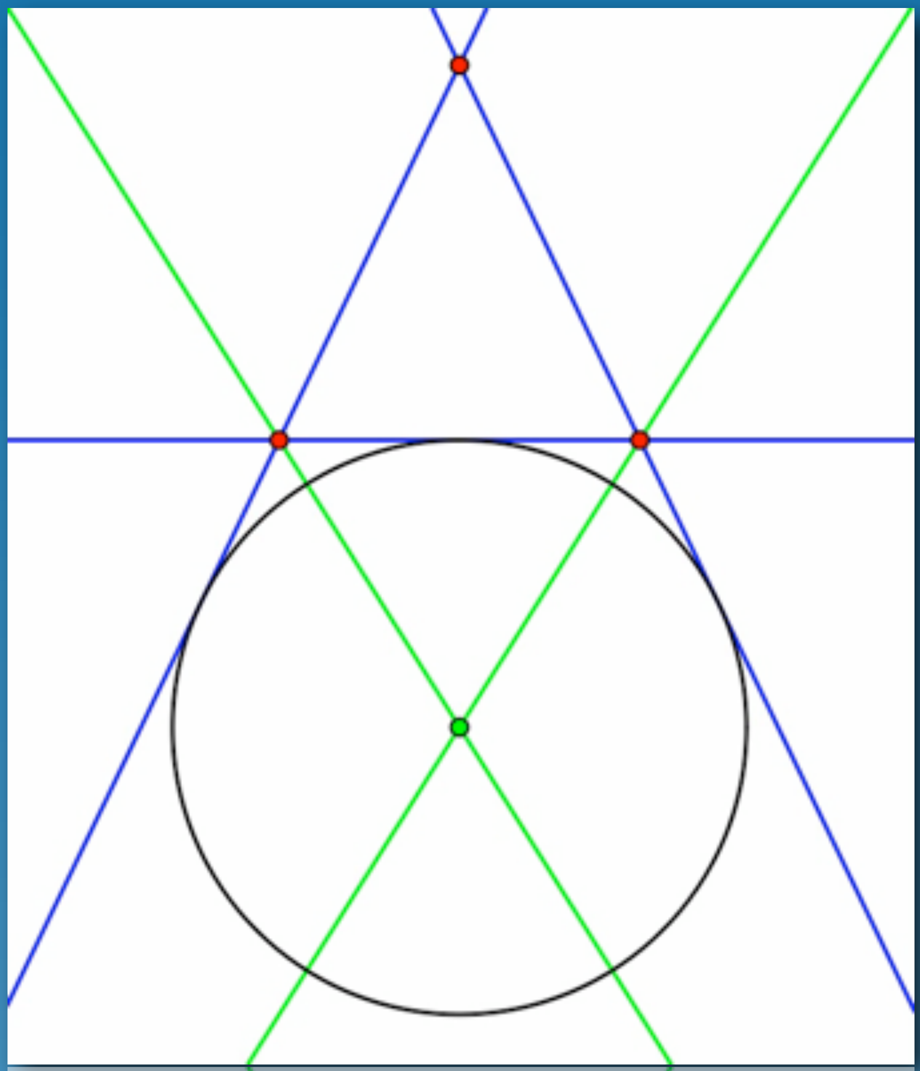

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- no lower bounds over $\mathbb{C}$

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    - constant point
    - join two points, meet two lines
    - intersection line and conic
    - intersection line and cubic curve

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considered operations	algebraically equivalent to
- constant point	constant
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goal: show this restriction is NP-hard

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Reach in dynamic geometry	→	*Reach* for GSPs

Considered operations

(performed at unit costs)

free	$\mathbb{C}$
constants	$\mathrm{const}_z = \{z\}$
arithmetic operations	$+ \;=\; \{(z_1, z_2, w) \in \mathbb{C}^3 \mid z_1 + z_2 = w\}$ $- \;=\; \{(z_1, z_2, w) \in \mathbb{C}^3 \mid z_1 - z_2 = w\}$ $* \;=\; \{(z_1, z_2, w) \in \mathbb{C}^3 \mid z_1 \cdot z_2 = w\}$ $/ \;=\; \{(z_1, z_2, w) \in \mathbb{C}^3 \mid z_1 = z_2 \cdot w \wedge z_2 \neq 0\}$
roots	$\sqrt{} \;=\; \{(z, w) \in \mathbb{C}^2 \mid z = w^2 \wedge z \neq 0\}$ $\sqrt[3]{} \;=\; \{(z, w) \in \mathbb{C}^2 \mid z = w^3 \wedge z \neq 0\}$

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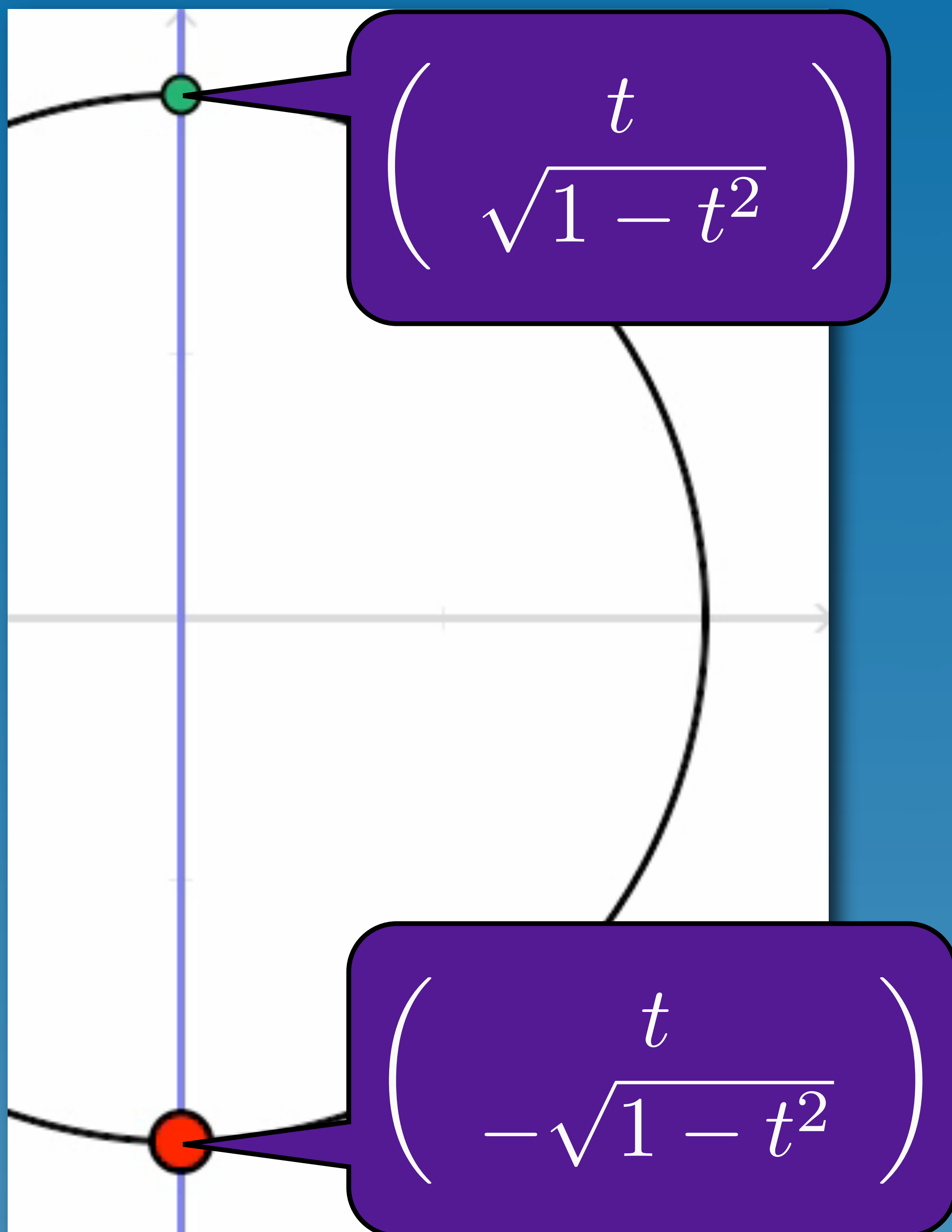

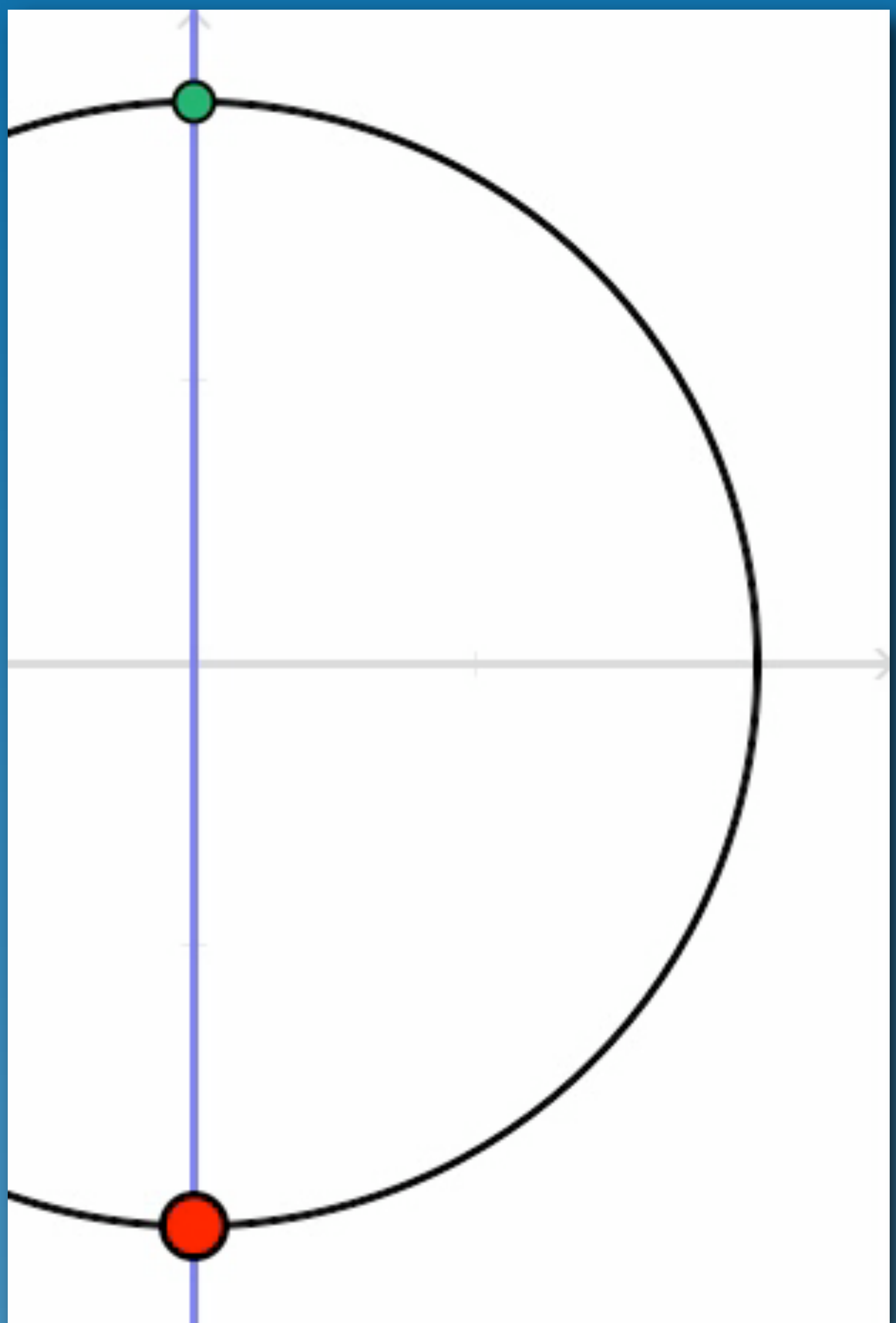

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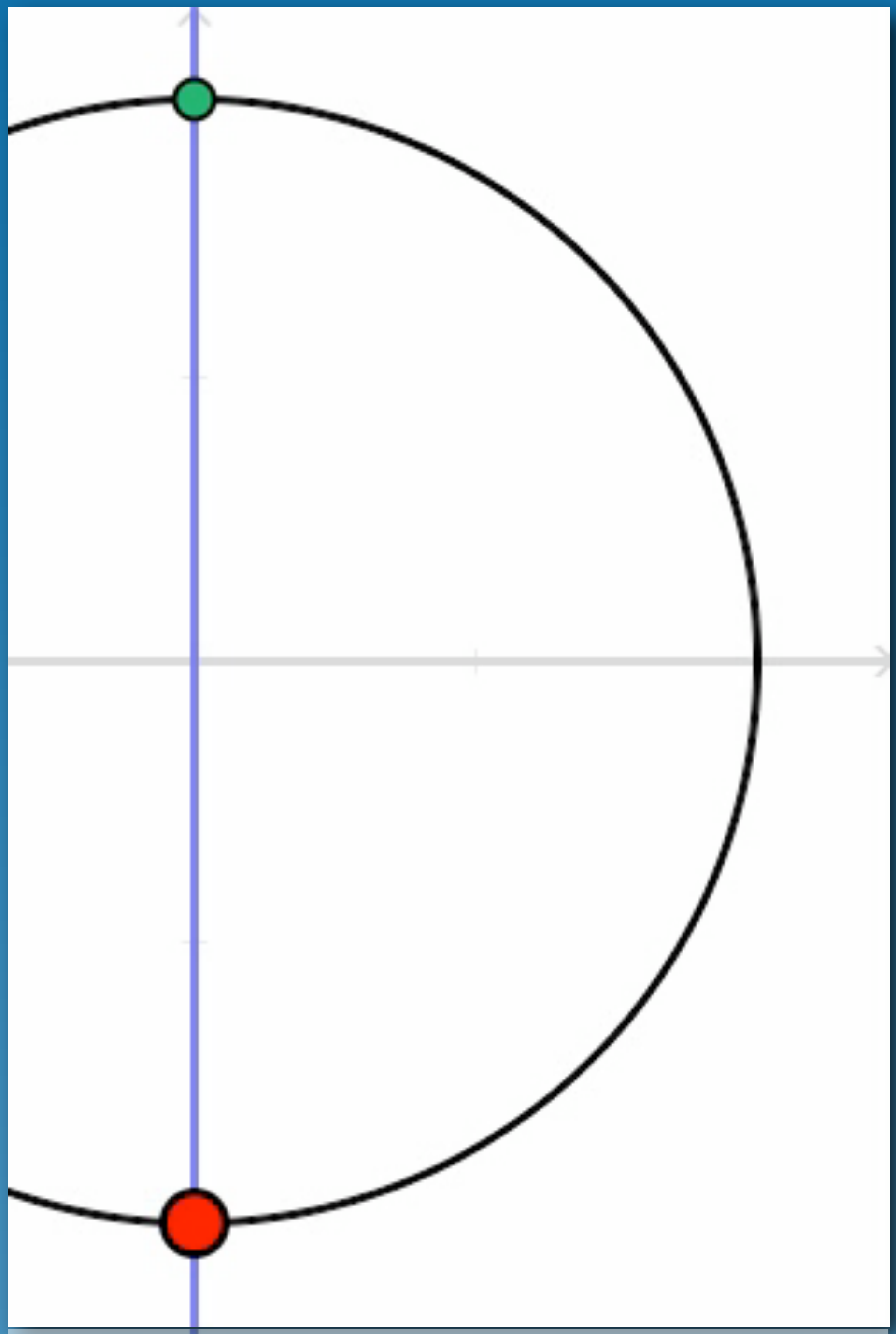

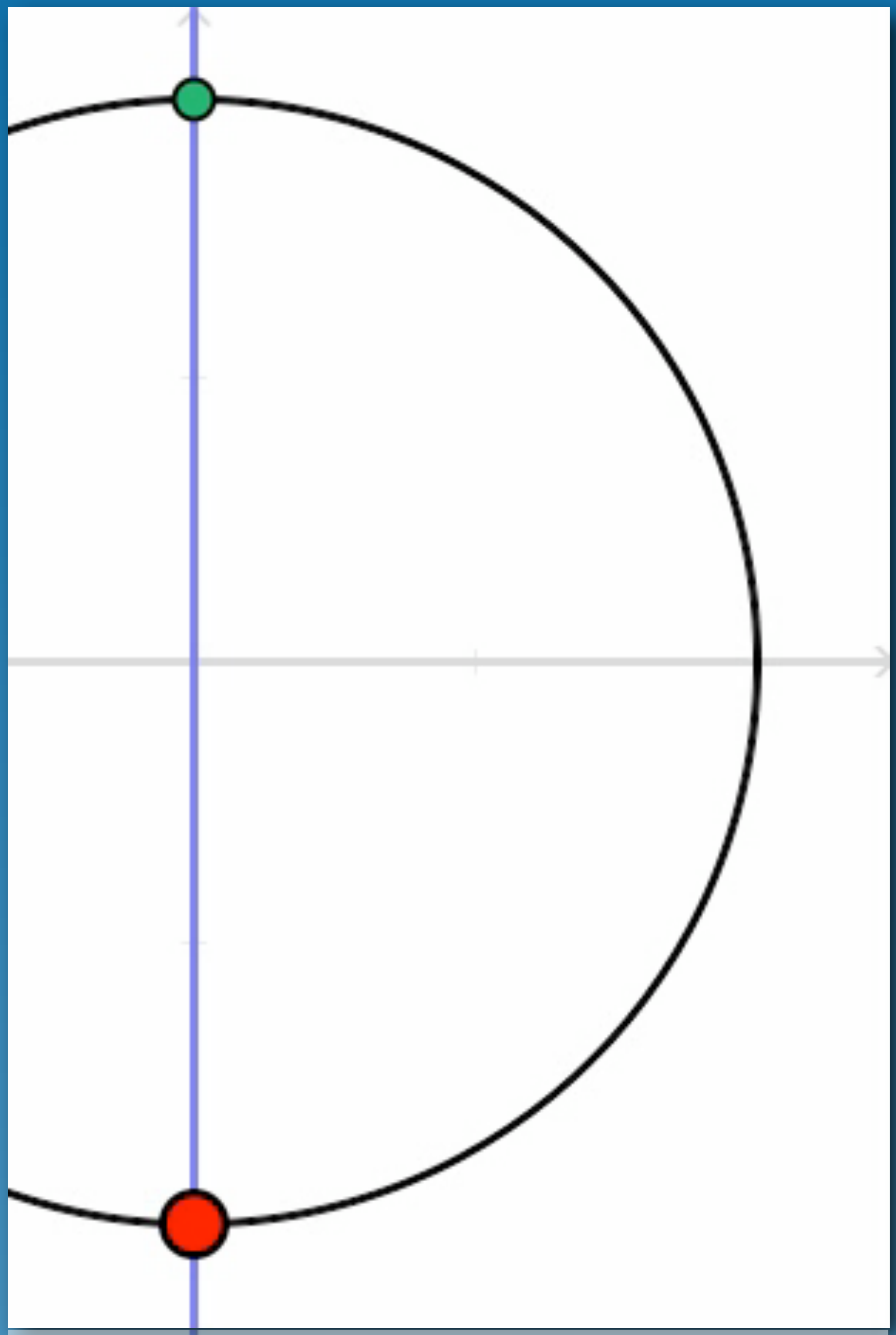

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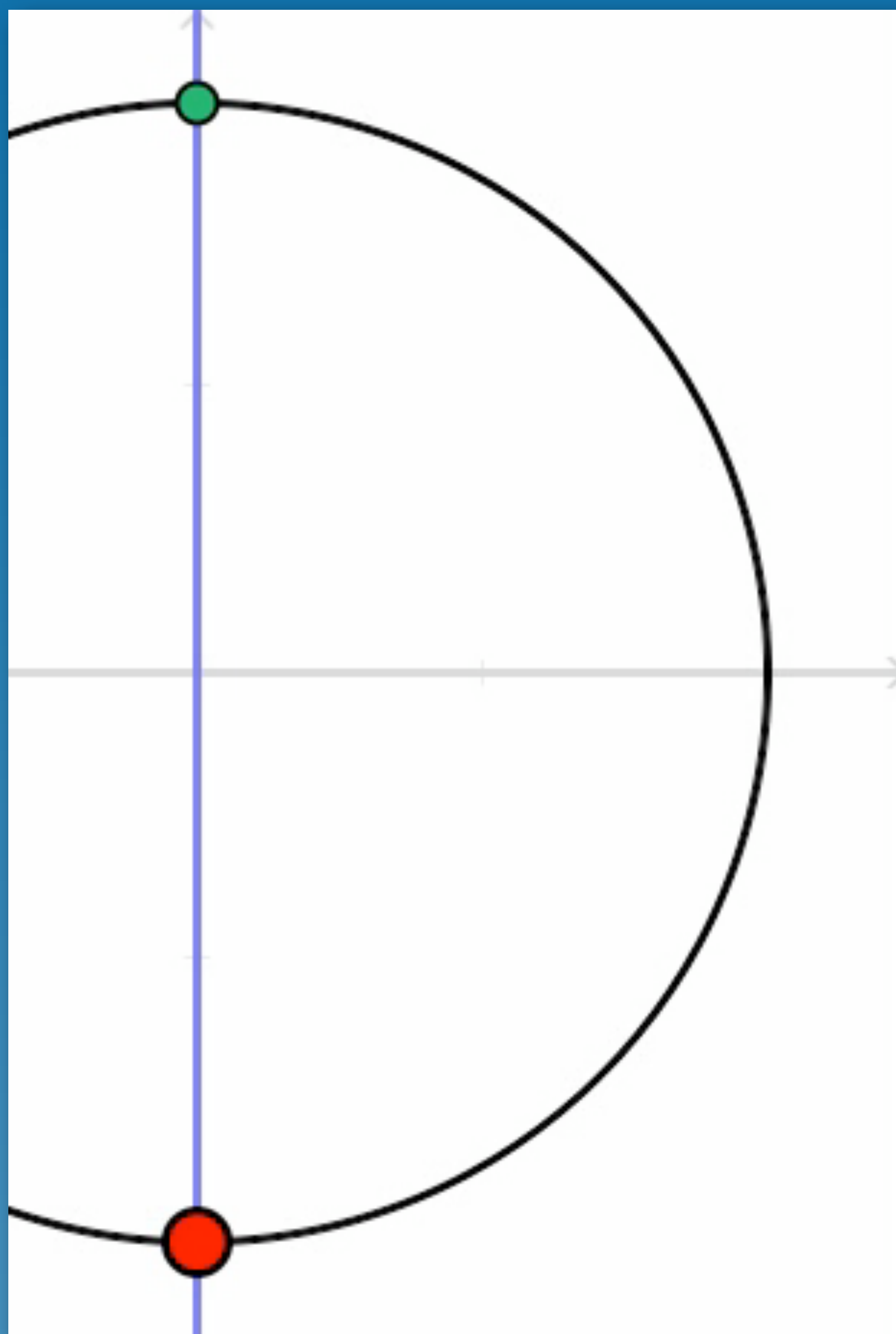

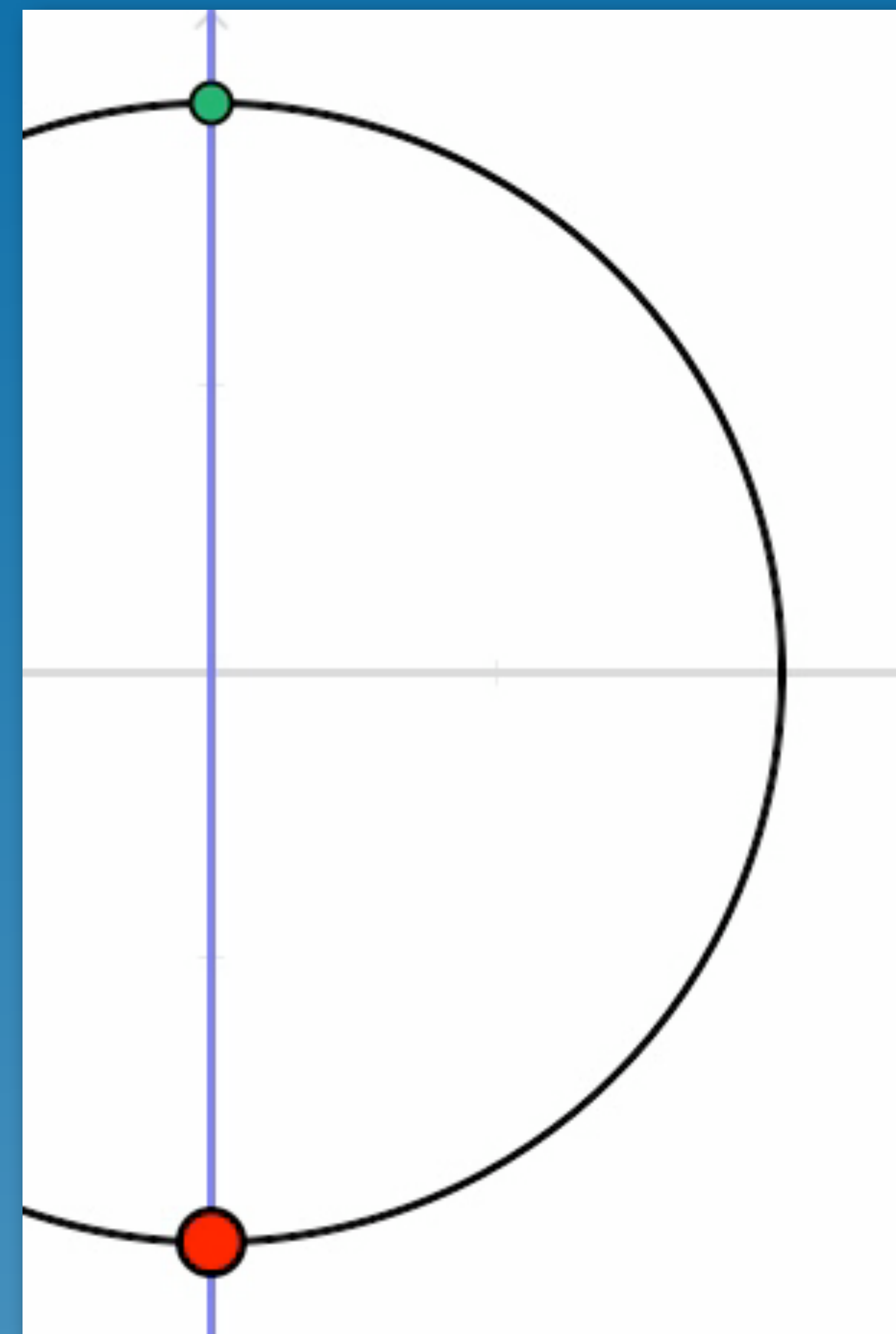

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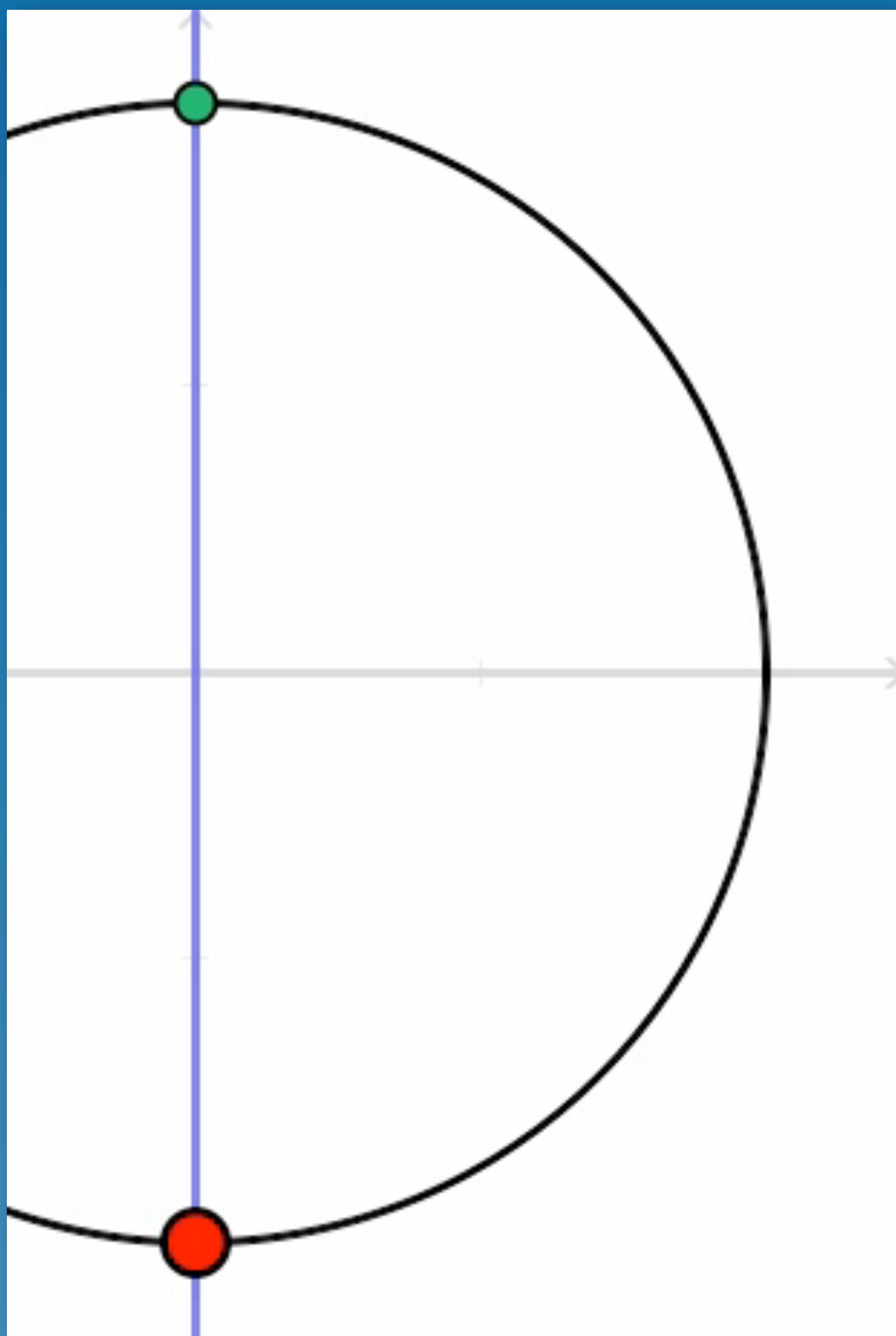

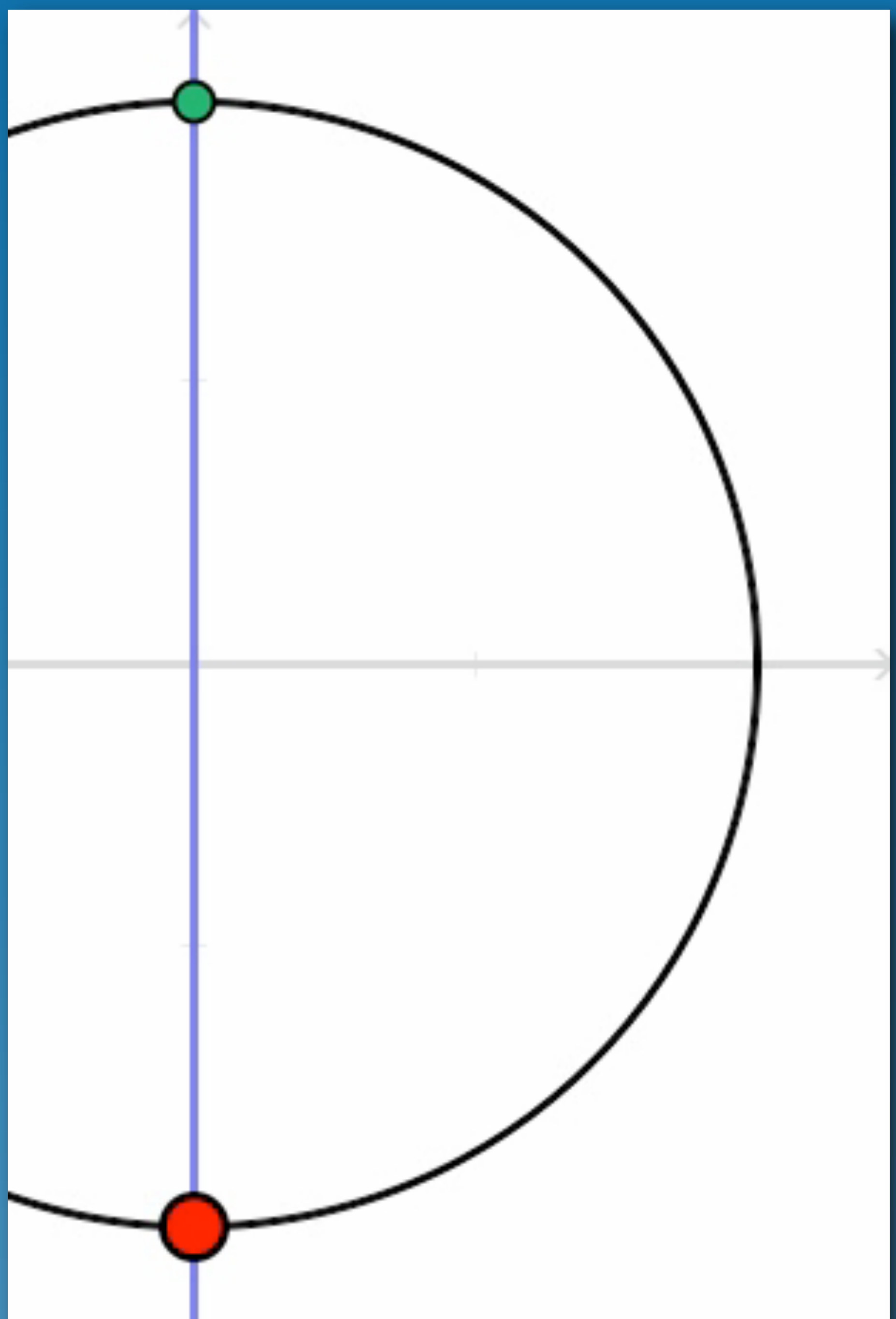

GSPs and their instances

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of operations with specified input assignments

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An **instance** of a GSP is an assignment of complex numbers

$$Z = (z_{1-p}, \ldots, z_p) \in \mathbb{C}^{p+q}$$

so that the relations of dependent operations $\omega_1, \ldots, \omega_q$ are satisfied

Reach - algebraic version

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given: GSP and two instance Z, W

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problem: Are there continous mappings

$$\mu_{1-p}, \ldots, \mu_0 : [0,1] \to \mathbb{C}$$
$$c_1, \ldots, c_q : [0,1] \to \mathbb{C}$$

so that

$$(\mu_{1-p}(t), \ldots, c_q(t))$$

forms an instance for all $t \in [0,1]$ and

$$(\mu_{1-p}(0), \ldots, c_q(0)) = Z \wedge (\mu_{1-p}(1), \ldots, c_q(1)) = W$$

A variant of 3SAT

bool. variables	$b_1, \ldots, b_n$
literals	$\{b_1, \ldots, b_n, \neg b_1, \ldots, \neg b_n\}$
clauses	$C_j = l_{j,r} \vee l_{j,s} \vee l_{j,t} \quad (l_{j,k} \in \{b_k, \neg b_k\})$
formula	$C = C_1 \wedge \ldots \wedge C_m$
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NP-complete

Outline reduction

1. From 3SAT formulas to functions
2. From truth assignments to analytic continuations
3. Counting multiplicities
4. Using the bounded length
5. Assembling the parts

Basic idea

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Transfer

$l_{j,k}$ literal	→	$x_{j,k}$ function
χ truth assignment	→	γ closed path

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Transfer

$l_{j,k}$ literal	⇨	$x_{j,k}$ function
χ truth assignment	⇨	γ closed path

so that

χ makes $l_{j,k}$

$true$	$\Longleftrightarrow$	$x^{\gamma}_{j,k}(0) = 0$
$false$	$\Longleftrightarrow$	$x^{\gamma}_{j,k}(0) = 1$

1. Formulas → Functions

$$\vee \quad \rightarrow \quad *$$

$$l_{j,k} = b_k \quad \rightarrow \quad x_{j,k}(z) = \frac{\sqrt{k} - \sqrt{k+z}}{2\sqrt{k}}$$

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$$C = C_1 \wedge \ldots \wedge C_m \quad \to \quad X(z) = \big(X_1(z), \ldots, X_m(z)\big)$$

2. Truth assign. → analy. continuations

$$\chi = (b_1, \ldots, b_n) \in \{true, false\}^n$$

↓ encode χ by the winding numbers of a closed path γ starting at $z = 0$

$$\eta(\gamma, -k) \in \begin{cases} 2\mathbb{Z}, & b_k = true \\ 2\mathbb{Z}+1, & b_k = false \end{cases}$$

Interplay $x_{j,k}$ and γ

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does not change ← $\eta(\gamma, -k) \in 2\mathbb{Z}$

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Interplay $x_{j,k}$ and γ

$$l_{j,k} = b_k$$

false

$$b_k = false$$

$$x_{j,k}(z) = \frac{\sqrt{k} - \sqrt{k+z}}{2\sqrt{k}}$$

does change

$$\eta(\gamma, -k) \in 2\mathbb{Z} + 1$$

$$x^{\gamma}_{j,k}(0) = 1$$

What have we achieved so far?

There is γ so that

$$X_C^\gamma(0) = (0, \ldots, 0) \in \mathbb{C}^m$$

$$\Updownarrow$$

C satisfiable

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Why we are not done?

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$$\Updownarrow$$

C satisfiable

Why we are not done?

specifying a terminal instance $\Longleftrightarrow$ knowing a satisfying truth assignment

3. Counting multiplicities

How many literals are true in a clause?

↓

How many factors vanish in $X_j^{\gamma}(0)$?

↓

idea: use the three branches of $\sqrt[3]{X_j(z)}$ as counters

3. Counting multiplicities

- set $Y_j(z) = \sqrt[3]{X_j(z)}$ (prinicipal branch)
- denote branches of cubic root by 0,1,2

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$$\Longrightarrow \; Y_j^{\gamma} \quad \text{lies on branch} \quad \frac{\eta(\gamma,0)}{2} \cdot M \pmod 3$$

nested roots

number of vanishing factors in $X_j^{\gamma}(0)$

3. Counting multiplicities

How to use it?

claim in starting instance

Y_j lies on branch 0

claim in terminal instance

Y_j^γ lies on branch 1

3. Counting multiplicities

How to use it?

claim in starting instance

Y_j lies on branch 0

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$$\frac{\eta(\gamma, 0)}{2} \cdot M \pmod 3 = 1$$

3. Counting multiplicities

How to use it?

claim in starting instance

Y_j lies on branch 0

claim in terminal instance

must control it

$$\frac{\eta(\gamma,0)}{2}\cdot M \pmod 3 = 1$$

4. Using bounded length

additional claims:

- $|\gamma| < 2(n+3) + \varepsilon\,,$ where $0 < \varepsilon \leq 1$
- γ circle around $-n-2$ and 1

5. Assembling the parts

exact 3SAT has answer **YES** $\iff$ $Y_1^\gamma, \ldots, Y_m^\gamma$ lie on branch 1

5. Assembling the parts

Idea proof:

1. adjusting multiplicities in X_j
2. switching $Y_j(z) = \sqrt[3]{X_j(z)}$ to branch I
3. reverse point I

Thank you for the attention!