WHAT IS A LINE?

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ADG 2010, 22-24 July, Munich, Germany

Thanks to the ADG organizers for the invitation.

2. The big picture

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I consider the projective geometry (Pappusian plane) as a reentrant Coq functor

input: a type of points, a type of lines, fulfiling Coxeter's axioms

output: an algorithmic theory which defines conics, etc, proves theorems, and generates new kind of lines: 3 times constrained conics.

3. Plane of the talk

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First part: the Pappus functor (nothing done up to now)

Second part: three times constrained conics are lines, the feature which makes the functor reentrant

4. First part: the Pappus functor

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The projective (Pappusian) plane is seen as a functor.

Input :

a type of points, a type of lines, which fulfils, say, Coxeter's axioms

algorithms to draw a line, join 2 points, intersect two lines

5. The Pappus functor outputs an algorithmic theory :

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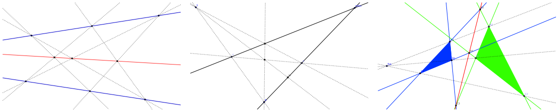
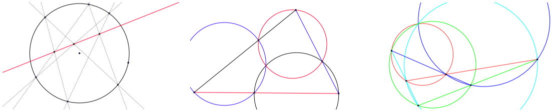
- definitions (conics),

- theorems and proofs: Pappus, harmonic conjugate, Desargue, ...



.. Pascal, 3-circles, 4-circles theorem



6. The Pappusian functor outputs also:

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- proved (extracted?) algorithms (draw a conic, ... a dynamic geometry software, an incidence prover based on matroids-hexamys,...)

- new objects: new points, new lines

7. The functor is reentrant

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The functor can be applied again on these new points and new lines.

it will generate new theorems (or extend existing ones) and new objects.

First time: conics are the usual ones (degree 2)

Second time: generated "conics" are cubics or quartics; in spite of their higher degree, they are still defined with 5 points.

8. Possible axioms for Pappus theory

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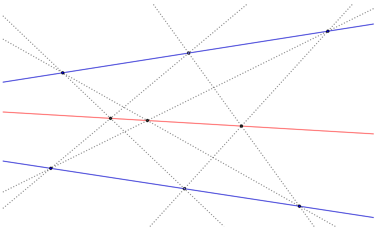
A1. 2 distinct points define exactly 1 line.

A2. 2 distinct lines meet in exactly 1 point (possibly at infinity).

A3 (Pappus). if p_1, p_2, p_3 alined, and q_1, q_2, q_3 alined, then r_{12}, r_{13}, r_{23} alined, with $r_{ij} = p_i q_j \cap p_j q_i$.



9. Possible axioms for Pappus theory: Coxeter's

Coxeter's axioms are a bit different, they rely on the definition of projectivity: a bijection between lines, defined by 3 pairs of points and their images. Equivalent to Pappus.

A4 (not characteristic 2) 4 points p_1, p_2, p_3, p_4, not 3 colinear: the 3 intersection points of the 6 lines are non colinear.

? A5. A projective plane with a complete quadrilateral exists.

Maybe a Coq program will need to explicit other axioms....

10. Why not using algebra and coordinates ?

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We don't want the theory to use facts like: lines are degree 1 curves, conics are degree 2 curves, because

- it will no more be true for non standard lines and conics

- and we want the Pappus functor to be reentrant

11. Why not using algebra and coordinates ?

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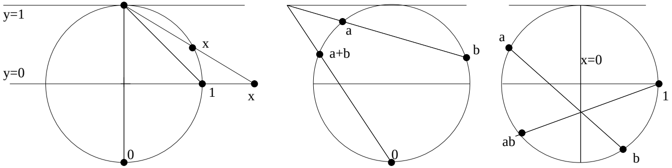
But, well, maybe it can work too?

And anyway we shall have to prove that a Pappus geometry defines a field (associativity, distributivity..).



12. From axioms, Pappus theory should define :

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projectivity (compositions of perspectivities) between 2 lines, between 2 bundles of lines

harmonic conjugate

homography

conics ...

13. Possible definition of conics

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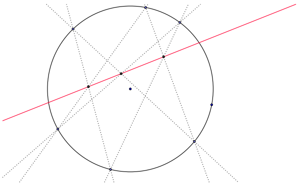
Via Pascal's theorem: given $p_1, \ldots p_5$, the locus of points p_6 s.t. opposite sides of the hexagon meet in 3 colinear points.

Hexamys theorem is Pascal in disguise:

An hexamys is an hexagon s.t. opposite sides meet in 3 colinear points.

All permutations of an hexamys are hexamys.



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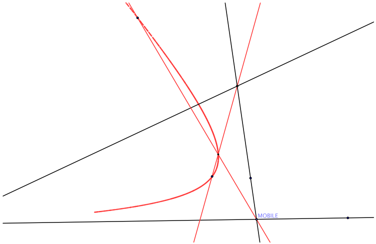
14. Another definition for conics :

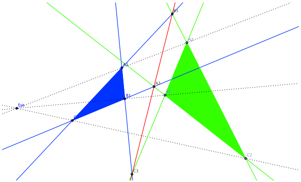


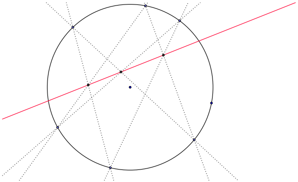
It is the locus of $l_i \cap l'_i$ where $l_i \in L, l'_i \in L', L$ and L' in projectivity.

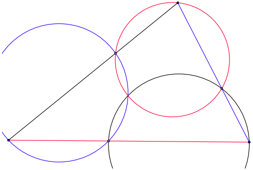
The theory should prove that all definitions of conics are equivalent.

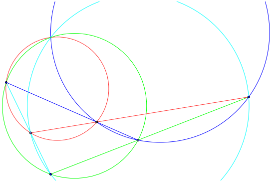
15. Then Pappus theory should prove theorems :

Desargue, Pascal, 3-circles, 4-circles, ...









16. Pappus theory should prove (extract?) algorithms

- given 2 conics and 2 intersection points, build the 2 others.

- draw a conic with Pascal's theorem

- prove forced incidences in figures, (say) with matroids and hexamys

- prove a dynamic geometry software...

17. Pappus theory has already been done

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several times, say by Vebber & Young, by Coxeter in "Projective Geometry".

But maybe not in a reentrant way ?

The goal of the game is to redo it in Coq, as a functor, and to do it in a reentrant way.

18. Second part

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Three times constrained conics are lines

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19. 3TCC are lines: sketch of proof

$\phi(x, y, h) = (x^2, y^2, h^2, xy, xh, yh)$

$\phi(x, y, h) \cdot Q = 0$ is a conic equation. Q lies in a 6d vector space.

A conic is defined by 5 points, Q is defined (up to its length) by 5 orthogonality conditions: consistent !

If Q is constrained to be orthogonal to 3 independent vectors (eg to pass through 3 points), then Q lies in a vector subspace with rank 6-3=3 : it is a line.

This sketch of proof does not use axioms of Pappus theory :(

btw, is it new ? likely no, but do you have a reference ?

20. Possible constraints on a conic vector Q

$C_1 = (1, -1, 0, 0, 0, 0), \quad C_2 = (0, 0, 0, 1, 0, 0).$

$\phi(\pm 1, i, 0) = (1, -1, 0, \pm i, 0, 0)$

$C_3 = (0, 0, 0, 0, 0, 1), \quad C_4 = 1, 0, -1, 0, 0, 1)$

$C_5 = (0, 1, 0, 0, 00), \quad C_6 = (1, 0, 0, 0, 0, 1)$

Q is a circle when $Q \cdot C_1 = Q \cdot C_2 = 0$. Or when $Q \cdot \phi(\pm 1, i, 0) = 0$. Center lies on line $y = 0$ if $Q \cdot C_3 = 0$. Circle is orthogonal to the unit circle if $Q \cdot C_4 = 0$.

C_6: condition for a circle to cut the unit circle in 2 symmetric points w.r.t. origin.

Q is a parabola with axis Oy if $Q \cdot C_2 = Q \cdot C_5 = 0$.

Being tangent to a fixed line does not give an orthogonality condition.

21. Examples of 3TCC, new lines

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- Circles through a fixed point : clines

- Circles with center on a fixed line (Poincaré half plane, a model for hyperbolic plane)

- Circles orthogonal to a fixed circle

- Circles cutting the unit circle in 2 points symmetric % origin

- Conics passing through 3 fixed (non colinear) points

- Parabolas with Oy axis and passing through a fixed point

22. Pappus theorem for clines

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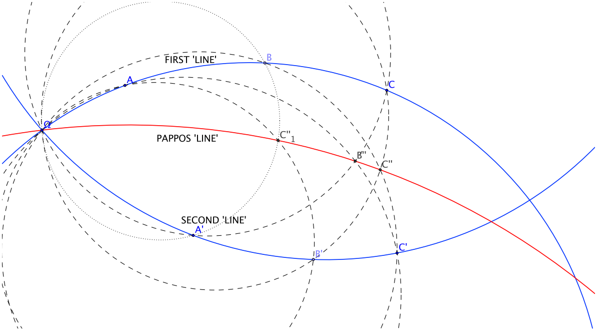

Clines fulfil Pappus property. Thus they can be considered as lines.

23. Harmonic conjugate theorem for clines

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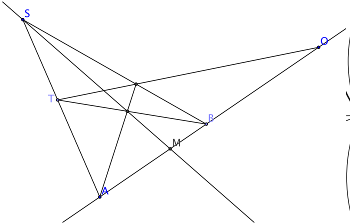

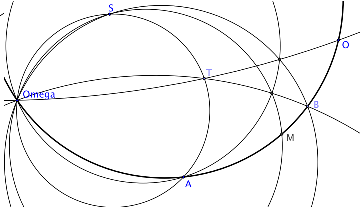

Left: for given points O, A, B on a common line, for any point S, for any point T on the line SA, the point M is invariant (hint: M is the harmonic conjugate of O relatively to A, B; if O is a point at infinity, M is the middle of AB.

Right: all lines are replaced with circles all passing through a fixed point. M is still invariant.

24. Pascal theorem for clines

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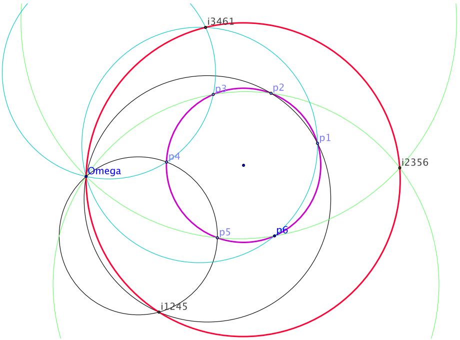

Points p_i lie on the magenta circle. The lines $p_i p_j$ are replaced with clines (circles through Omega).

The intersection points lie on a common cline (red circle).

25. Desargue for clines

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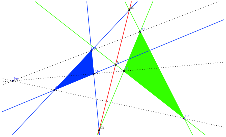

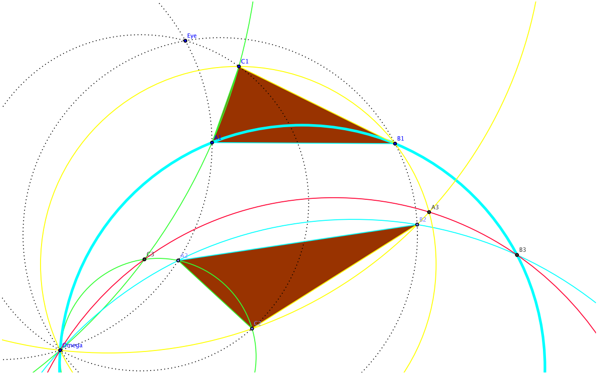

26. Poincaré half plane is Pappusian

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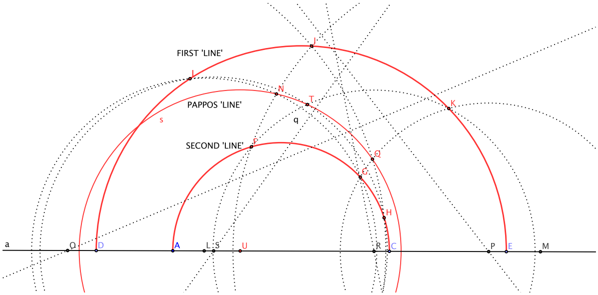

27. Conics through 3 fixed points fulfil Pappus

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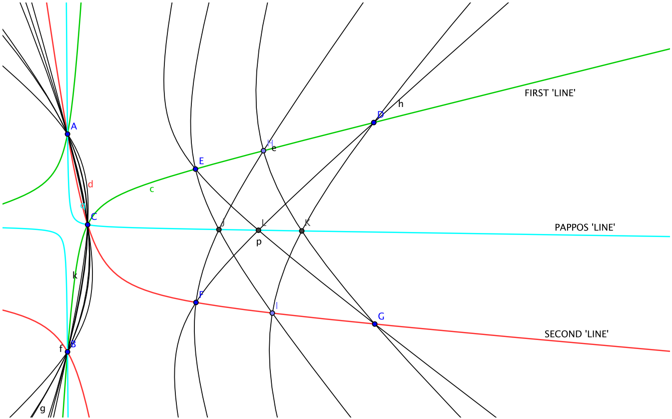

Conics through 3 fixed points fulfil Pappus, thus they are lines.

28. Harmonic conjugate theorem

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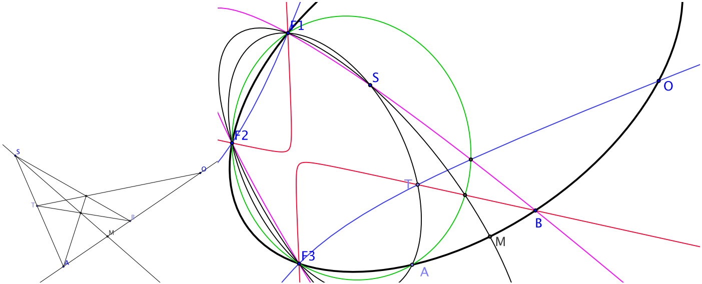

Left: the harmonic conjugate theorem for naive lines.

Right: the harmonic conjugate theorem for conics passing through 3 fixed points F_1, F_2, F_3.

29. In passing, a regret : tropical lines are not lines

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Jürgen Richter-Gebert et al show that tropical lines do not always fulfil Pappus property.

It is a pity, because otherwise, we could enjoy (rational) tropical witnesses for Geometric Constraints Solving...

The study of the witness gives informations on the system of geometric constaints to solve.

30. Lines-circles-conics should be extended consistently!

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In theory, Pascal's theorem gives the answer:

the conic through $p_1, \ldots p_5$ is the locus of points p_6 s.t. opposite "lines" $p_1 p_2$ and $p_4 p_5$, $p_2 p_3$ and $p_5 p_6$, $p_3 p_4$ and $p_6 p_1$ meet in 3 points on the same "line".

A Pappus functor in Coq should help us to consistently generalize lines, circles, conics.

Example : 3 circles theorem.

31. Example of a theorem, 3 circles

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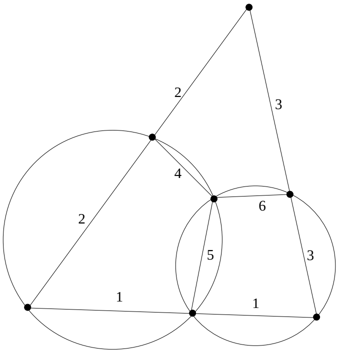

Points $1 \cap 2, 2 \cap 4, 4 \cap 5, 5 \cap 1$ are cocylic, as well as $5 \cap 6, 6 \cap 3, 3 \cap 1, 1 \cap 5$. We need to prove that the points $2 \cap 3, 3 \cap 6, 6 \cap 4$, $4 \cap 2$ are cocyclic too.

Note 1, 2,... the orthogonal symmetry rel. to line 1, 2, ...

Lemma: 5124 is "cocyclic" $\Rightarrow$ 5124 is a translation

32. Example of a theorem, 3 circles, proof 1

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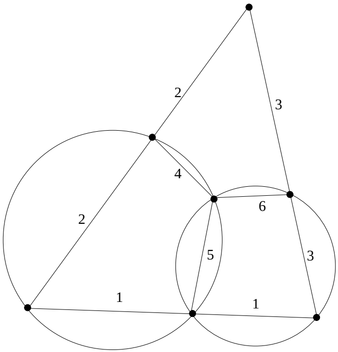

Note 1, 2,. . . the orthogonal symmetry rel. to line 1, 2, . . .
Lemma: 5124 is "cocyclic" $\Rightarrow$ 5124 is a translation
Idem: 6315 is a translation.
Thus (6315)(5124) = 6324 is a translation, thus is "cocyclic".
Pretty proof, but we can not replace lines with 3TCC :(

33. Example of a theorem, 3 circles, degenerate case

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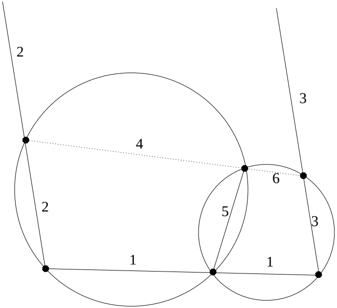

We need to prove line 4= line 6. As before: 6315 and 5124 are translations. Thus (6315)(5124) = 6324 as well. Thus 3246 as well. Moreover 32 and 23 are translations (2 // 3). Thus (23)(3246) = 46 is a translation. Thus 4//6. But they have a common point ($6 \cap 5$ and $4 \cap 5$), thus they are equal. QED.

34. 3 circles theorem, proof 2

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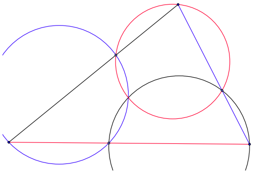

The 3 cubics :

- circle $AB'C'$ + line $A'BC$

-circle $A'BC'$ + line $AB'C$

-circle $AB'C'$ + line $A'BC$

have 8 common points $ABCA'B'C'$ and the two cyclic points
Thus (Chasles' theorem) they have a 9 th common point.
This proof does not use Pappus axioms :(

35. Extension of 3 circles theorem

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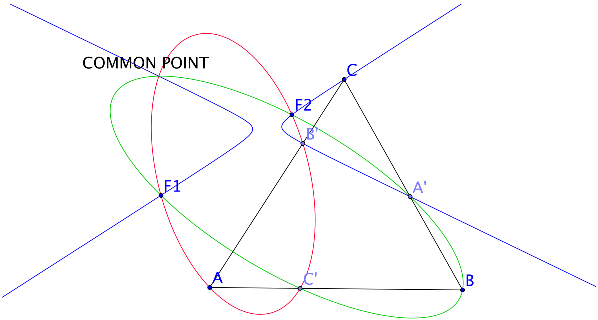

Circles are replaced with conics passing through 2 distinct arbitrary points. These 3 conics have a common point (other than the 2 arbitrary points).

We want to replace standard lines with non standard lines.

36. A generalization of 3-circles which works:

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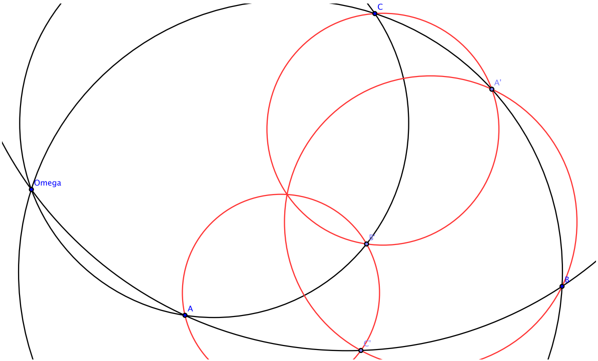
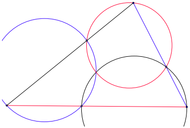


Lines can be replaced with clines.

Proof: just perform an inversion on the standard figure.



37. A generalization which does not work:

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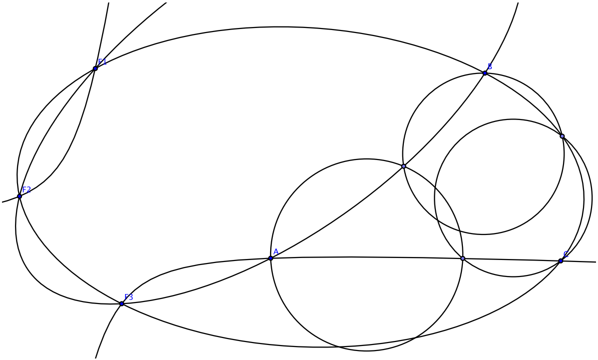

Replacing lines with conics passing through 3 fixed points, and circles with circles, is not consistent.

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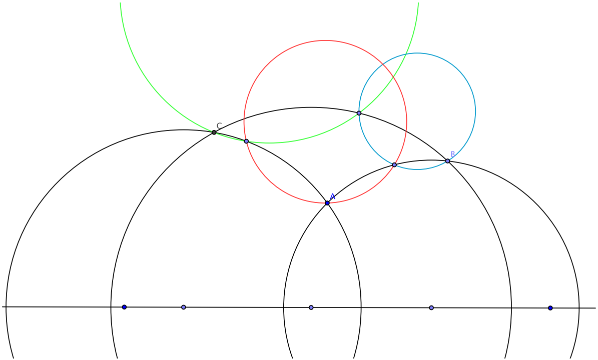
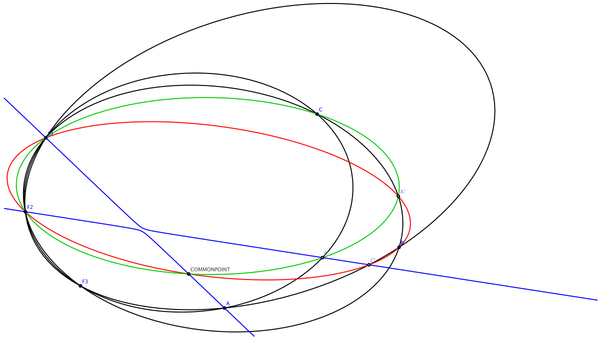
38. A 2nd generalization which does not work:



Replacing lines with cercles with centers on a fixed line, and replacing circles with circles, is not consistent either.

A Coq implementation should give an automatic method for consistent generalizations of lines-circles-conics.

39. A non trivial generalization which works:



Lines are replaced by conics passing through 3 fixed points F_1, F_2, F_3

Circles are replaced by conics passing through F_1, F_2.

These 3 "circles" have a common point.

40. 4 circles theorem, standard case, proof 1

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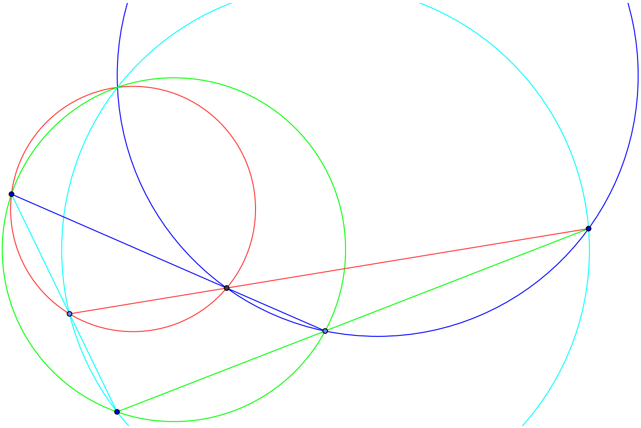

Proof: the 4 cubics (a circle + "opposite" line) meet in 8 points (2 cyclic points and 6 points of the complete quadrilateral). Thus they meet in a 9nth point (Chasles' theorem).

41. 4 circles theorem, standard case, proof 2

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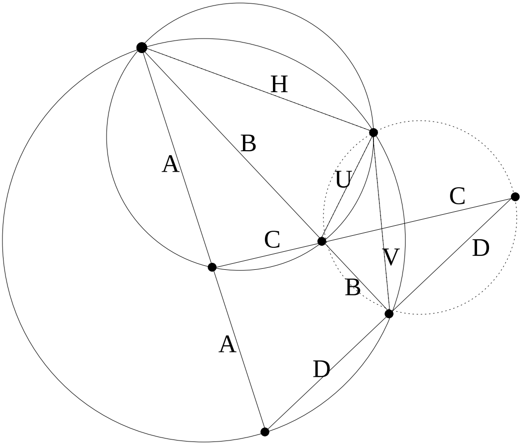

By hypothesis, $ACUH$ is cocyclic, $ACUH$ is a translation. $HVDA$ as well. Thus $(ACUH)(HVDA) = ACUVDA$ is a translation, thus $CUVDAA = CUVD$ as well. Thus $CUVD$ is cocyclic. QED. Pretty proof, but does not belong to Pappus theory :(

41. Extension of the 4 circles theorem

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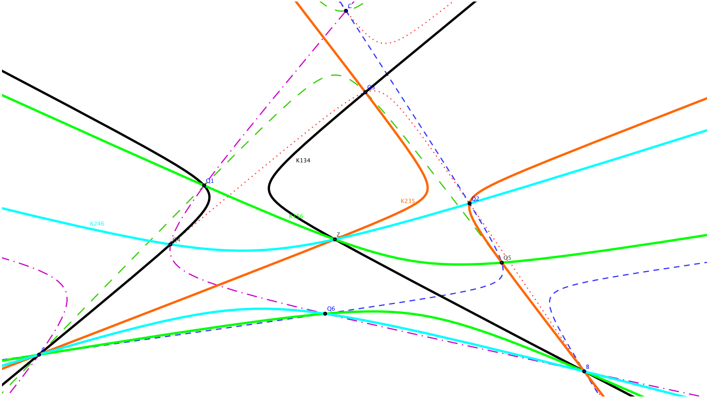

Lines are replaced with conics through points A, B, C.
(A, B, C, Q_1, Q_3, Q_5), (A, B, C, Q_1, Q_4, Q_6), (A, B, C, Q_2, Q_3, Q_4), (A, B, C, Q_2, Q_5, Q_6) are "aligned".

Circles are replaced by conics through A, B. The 4 "circles" $K134(A, B, Q_1, Q_3, Q_4)$, $K156$, $K235$, $K246$ meet in Z.

43. Unnamed theorem: standard case

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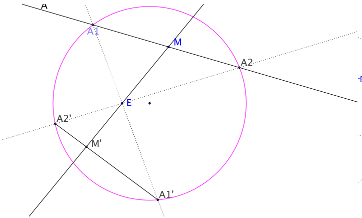

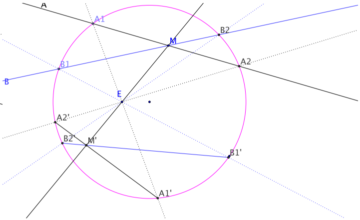

M' is the "symmetric" of M r.t. E. It does not depend on the auxilliary chord A_1A_2, B_1B_2.

44. Unnamed theorem: standard / clines

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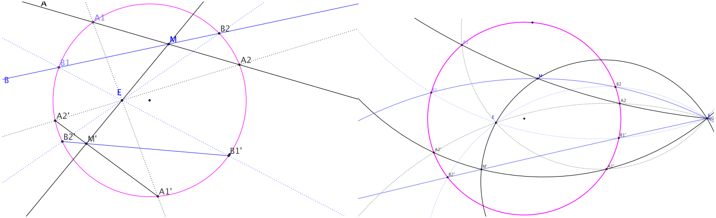

45. Unnamed theorem: with conics through F_1, F_2, F_3

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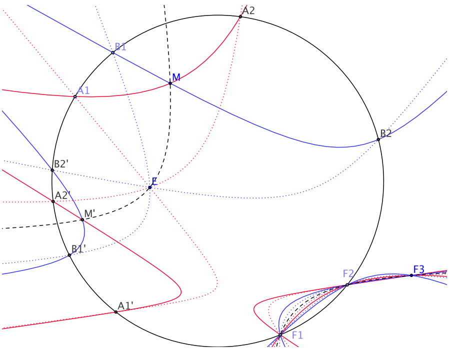

M' does not depend on the used chord.

Lines are replaced with conics through F_1, F_2, F_3. The circle is replaced with a circle through F_1, F_2. (It could be replaced with a conic through F_1, F_2.)

46. Conclusion: A Pappus functor should permit

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- to generate an infinity of new objects (non standard lines), and new theorems

- to consistently extend lines-circles-conics

- to extract a dynamic geometry software handling these new objects

- to generate a prover (hexamys + matroids ?) of forced incidences in figures

Pappus theory has already been formalized (Vebber & Young, Coxeter), but not in Coq, not in a reentrant way.

Similar with bootstrap compilers.

Nothing done yet !