Readable Machine Proofs for Mass Point Geometry

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July 24, 2010, Munich

Outline

1. Mass Point Geometry

Readable Machine Proofs for Mass Point Geometry
—— Hilbert Intersection Point Statements (CH)

2. Complex Coefficient Mass Point Geometry

Readable Machine Proofs for Mass Point Geometry
—— Linear Constructive Geometry Statements (CL)

Mass Point Geometry

1. *Mass Point Geometry Preliminaries*
2. *Examples and description of Mass-Point Method*
3. *Algrithm MPM*
4. *Implementation of algorithm MPM*
5. *Running examples*

Mass Point Geometry Preliminaries

$m \to \bullet P$

mass point

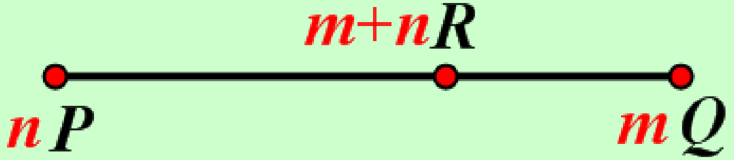
$mP(m \neq 0, m \in \mathrm{R})$

$xM \bullet\!\!\longrightarrow\!\!\bullet\, xN$

vector $x\,\overline{MN}$

$xM - xN(x \in \mathrm{R})$

Crucial idea:



$nP + mQ = (n+m)R \quad (m+n \neq 0)$

where

(i) R is on $\overline{PQ}$ and

(ii) $\overline{PR} : \overline{RQ} = m : n$;

$\Longleftarrow$

$\overline{PR} : \overline{RQ} = m : n$

$\Rightarrow n\,\overline{PR} = m\,\overline{RQ}$

$\Rightarrow nP - nR = mR - mQ$

$\Rightarrow nP + mQ = (n+m)R$

let $u = \dfrac{n}{m+n}$

$\Rightarrow R = (1-u)P + uQ$

Basic Propositions in Mass Point Geometry

Proposition B.1 *Let* $m+n\neq 0$, *then* $(m+n)C = nA + mB \Leftrightarrow C \in l_{AB}$ *and* $\overline{AC}:\overline{CB} = m:n$.

Proposition B.2 $O = AB \bigcap PQ \Leftrightarrow$ *There exists* $x, y, m, n \in \mathrm{R}$ *satisfying*

$x + y = m + n = t \neq 0$ *such that* $xA + yB = tO = mP + nQ$.

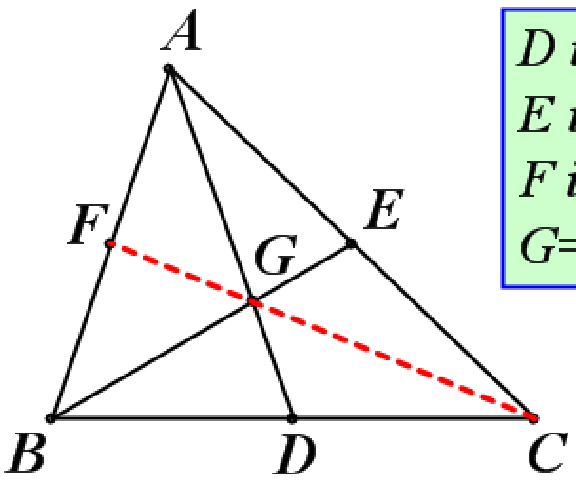
Proposition B.3 $x\overline{AB} = y\overline{PQ}$ $(x, y \neq 0, AB \parallel PQ) \Leftrightarrow xA - xB = yP - yQ$.

Proposition B.4 *Any point* P *on plane* ABC *could be expressed as the linear combination of* A, B, C *uniquely, that is* $P = aA + bB + (1-a-b)C$.

Proposition B.5 *Let* P, Q *and* R *be points on plane* ABC, *and* $P = aA + bB + (1-a-b)C$, $Q = xA + yB + (1-x-y)C$, $R = sA + tB + (1-s-t)C$, *then* $\dfrac{S_{PQR}}{S_{ABC}} = \begin{vmatrix} a & b & 1-a-b \\ x & y & 1-x-y \\ s & t & 1-s-t \end{vmatrix}$.

Proposition B.6 P, Q *and* R *are collinear if and only if* $S_{PQR} = 0$.

A Simple Example: Theorem of Centroid



D is the midpoint of BC;
E is the midpoint of AC;
F is the midpoint of AB;
G=AD∩BE;

$\Rightarrow$

C,G and F are collinear;

$$\frac{\overline{AG}}{\overline{GD}} = \frac{\overline{BG}}{\overline{GE}} = \frac{\overline{CG}}{\overline{GF}} = 2$$

$$2D = B + C$$
$$2E = A + C$$
$$2F = A + B$$
$$G = AD \cap BE$$

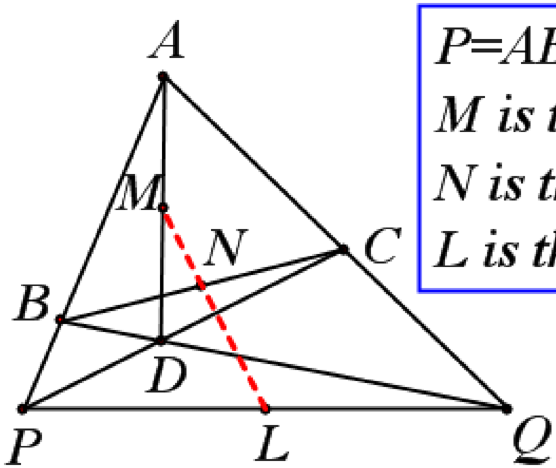
$$A + 2D = 3G$$
$$B + 2E = 3G$$
$$C + 2F = 3G$$

Mass Point Method:

$$\left.\begin{aligned} 2D &= B + C \\ 2E &= A + C \end{aligned}\right\} \xrightarrow[C]{eliminate} A + 2D = B + 2E \left(\xrightarrow[G=AD\cap BE]{further} \right) = 3G$$

$$\left.\begin{aligned} 3G &= A + B + C \\ 2F &= A + B \end{aligned}\right\} \xrightarrow[A,B]{eliminate} 3G = 2F + C$$

Example 2: Gauss-line Theorem



P=AB∩CD; Q=AC∩BD;
M is the midpoint of AD;
N is the midpoint of BC;
L is the midpoint of PQ;

$\Longrightarrow$

A,B,C
D=aA+bB+(1-a-b)C
P=AB∩CD; Q=AC∩BD;
2M=A+D; 2N=B+C;
2L=P+Q

$\overset{?}{\Longrightarrow}$ ***M, N and L are collinear***

$$D = aA + bB + (1-a-b)C$$

$$\Rightarrow \begin{cases} D-(1-a-b)C = aA+bB\left(\xrightarrow[P=AB\cap CD]{further}\right) = (a+b)P \\ D-bB = aA+(1-a-b)C\left(\xrightarrow[Q=AC\cap BD]{further}\right) = (1-b)Q \end{cases}$$

$$\Rightarrow \begin{cases} 2L = P+Q \xrightarrow{further} 2L = \dfrac{a(a+1)}{(a+b)(1-b)}A + \dfrac{b}{a+b}B + \dfrac{1-a-b}{1-b}C \\ 2M = A+D \xrightarrow{further} 2M = (a+1)A + bB + (1-a-b)C \end{cases}$$

$$2N = B + C$$

$$\Rightarrow 2(a+b)(1-b)L - 2aM = 2b(1-a-b)N \quad \textbf{or} \quad S_{LMN} = 0$$

Mass Point Method

1. *Formulate the geometry theorem in a constructive way and state the conclusion(s) of the theorem as a(an) (in)definite mass-point expression (expressions).*

2. *Interpret the constructions into mass-point expressions, try to eliminate the points from the existing mass-point relations according to the constructing requirements, and keep gaining new mass-point relations until the end construction.*

3. *Check whether the conclusions are included in the existing mass-point expressions; or continue to derive from the existing mass-point expressions and get the required results.*

Description of the Mass Point Method

1. The Constructions of the Hilbert Intersection Point Statements (CH)

$Point(X)$: Take an arbitrary point X on a plane.

$Points(A,B,C): Point(A)+Point(B)+Point(C);$

$Aratio(X,A,B,C,a,b): X = aA+bB+(1-a-b)C;$

$Lratio(X,A,B,r)$: $\overline{AX} = r\overline{AB}\left(X = (1-r)A+rB\right)$

$Mratio(X,A,B,r): \overline{AX} = r\overline{XB} \quad ((1+r)X = A+rB);$

$Midpoint1(X,A,B): 2X = A+B;$

$Pratio(X,U,A,B,r)$: $\overline{UX} = r\overline{AB} \quad \left(X = U + r(B-A)\right)$

$Parallel(X,U,A,B): \overline{UX} = \overline{AB} \quad \left(X = U+B-A\right);$

$Inter(X,U,V,A,B)$: $X = UV \cap AB$

$Pinter(X,A,B,W,U,V)$: $Parallel(Y,W,U,V)+Inter(X,W,Y,A,B)$

$PPinter(X,C,A,B,W,U,V)$:

$Parallel(Y,W,U,V)+Parallel(Z,C,A,B)+Inter(X,W,Y,C,Z)$

Description of the Mass Point Method

2. The Form of Conclusion in CH: problem-solving objevtives

$ifparallel(X,Y,A,B)$	*Check whether XY is parallel to AB;*
$ifmidpoint(C,A,B)$	*Check whether C is the midpoint of AB;*
$ifcolinear(A,B,C)$	*Check whether A, B and C are collinear;*
$ifequal(X,Y)$	*Check whether X coincides with Y;*
$ratio(A,B,X,Y)$	*Find the ratio of two parallel segments AB and XY;*
$arearatio(A,B,C,X,Y,Z)$	*Find the ratio of two signed area of oriented triangles ABC and XYZ.*

Algorithm MPM *(Mass-Point-Method)*

A constructive statement (S) in class CH:
constructions $\left(C_1, C_2, \cdots, C_k\right)$ + *objectives* $\left(E_1^*, E_2^*, \cdots\right)$

$$S = \left(C_1, C_2, \cdots, C_k, G = \left(E_1^*, E_2^*, \cdots\right)\right)$$

MPM

- **CF** — *construction function set*
- **AF** — *objectives function set*
- **MMC** — *controller*

Implementation of Algorithm MPM

We needs three global variables to implement MPM:

pnts *records the number of points*

number of points

varlist *records all the points introduced*

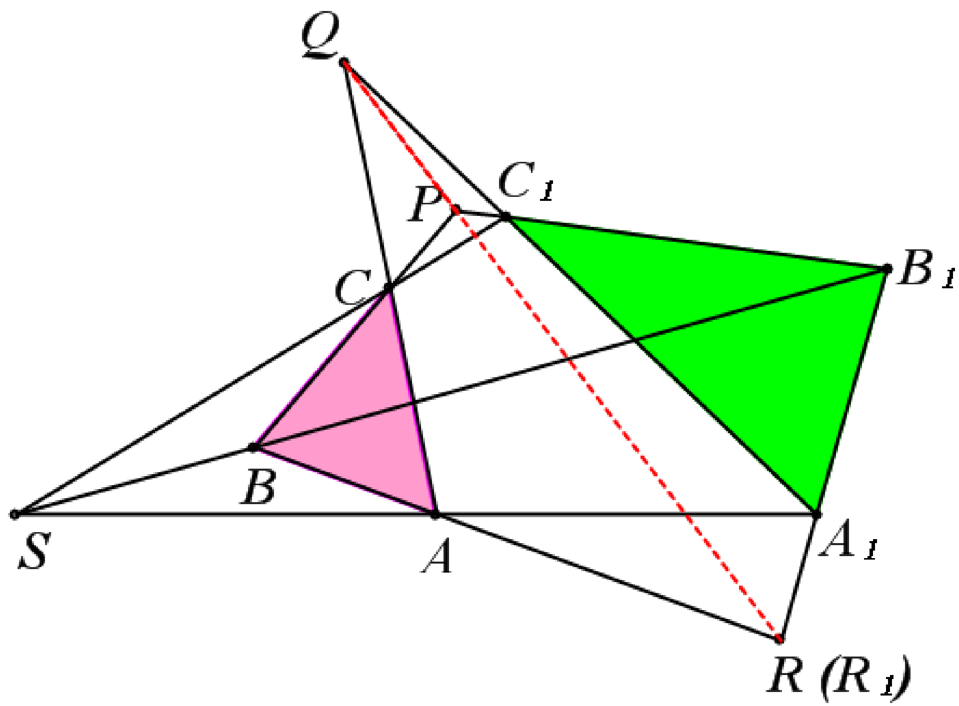
point list : an ordered set where the location of each point is fixed.

coordlist *records all the corresponding vectors of points*

vector list: an ordered set whose element is corresponding with point in varlist.

Running example1:

Desargues' Theorem




```
Points(A, B, C);
Aratio(S, A, B, C, a, b);
Mratio(A1, S, A, x);
Mratio(B1, S, B, y);
Mratio(C1, S, C, z);
Inter(P, B, C, B1, C1);
Inter(Q, A, C, A1, C1);
Inter(R, A, B, A1, B1);
Inter(R1, A, B, P, Q);
ifequal(R, R1);
```

Machine Proof for Desargues' Theorem

$$A,\ B,\ C$$

$$S = a\ A + b\ B + (1 - a - b)\ C$$

$$(1 + x)\ A1 = S + x\ A$$

$$(1 + y)\ B1 = S + y\ B$$

$$(1 + z)\ C1 = S + z\ C$$

$$P = (B\ C) \cap (B1\ C1)$$

$$B - \frac{z\ C}{y} = \frac{(1 + y)\ B1}{y} - \frac{(1 + z)\ C1}{y}$$

$$P = \frac{y\ B}{y - z} - \frac{z\ C}{y - z}$$

$$Q = (A\ C) \cap (A1\ C1)$$

$$A - \frac{z\ C}{x} = \frac{(1 + x)\ A1}{x} - \frac{(1 + z)\ C1}{x}$$

$$Q = \frac{x\ A}{x - z} - \frac{z\ C}{x - z}$$

$$R = (A\ B) \cap (A1\ B1)$$

$$A - \frac{y\ B}{x} = \frac{(1 + x)\ A1}{x} - \frac{(1 + y)\ B1}{x}$$

$$R = \frac{x\ A}{x - y} - \frac{y\ B}{x - y}$$

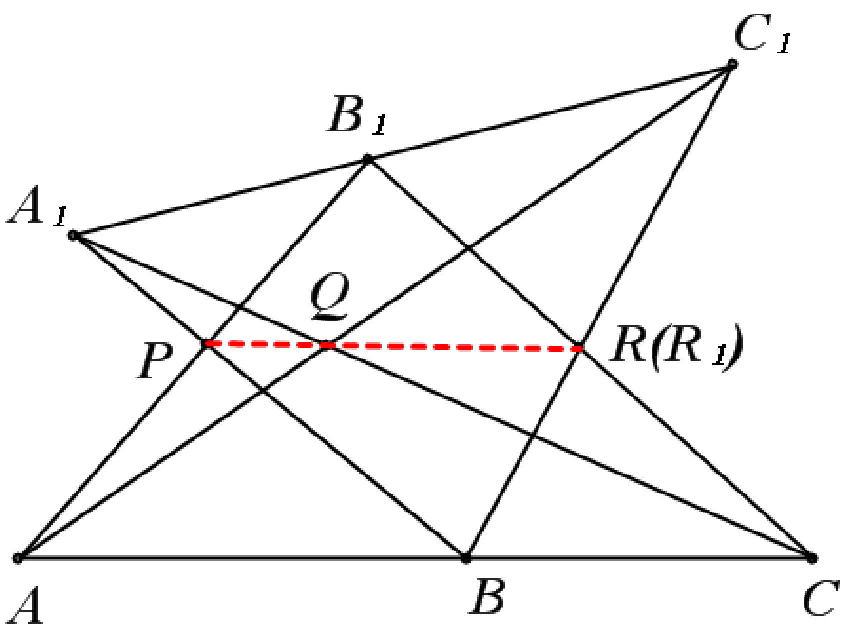
$$R1 = (A\ B) \cap (P\ Q)$$

$$A - \frac{y\ B}{x} = -\frac{(y - z)\ P}{x} + \frac{(x - z)\ Q}{x}$$

$$R1 = \frac{x\ A}{x - y} - \frac{y\ B}{x - y}$$

$$R = R1$$

Running example2: **Pappus' Theorem**




```
Points(A, B, A1);
Aratio(B1, A, B, A1, a, b);
Mratio(C, A, B, x);
Mratio(C1, A1, B1, y);
Inter(P, A1, B, A, B1);
Inter(Q, A1, C, A, C1);
Inter(R, B, C1, B1, C);
Inter(R1, B, C1, P, Q);
ifequal(R, R1);
```

Machine Proof for Pappus' Theorem

$$A,\ B,\ A1$$

$$B1 = a\ A + b\ B + (1 - a - b)\ A1$$

$$(1+x)\ C = A + x\ B$$

$$(1+y)\ C1 = A1 + y\ B1$$

$$P = (A\ B1)\ \cap\ (A1\ B)$$

$$A1 - \frac{b\ B}{-1+a+b} = \frac{a\ A}{-1+a+b} - \frac{B1}{-1+a+b}$$

$$P = -\frac{b\ B}{-1+a} + \frac{(-1+a+b)\ A1}{-1+a}$$

$$Q = (A\ C1)\ \cap\ (A1\ C)$$

$$A1 - \frac{y\ b\ (1+x)\ C}{x\ (-1-y+y\ a+y\ b)} = \frac{(a\ x-b)\ y\ A}{x\ (-1-y+y\ a+y\ b)} - \frac{(1+y)\ C1}{-1-y+y\ a+y\ b}$$

$$Q = -\frac{y\ b\ A}{y\ a\ x - x\ y - x - y\ b} - \frac{y\ b\ x\ B}{y\ a\ x - x\ y - x - y\ b} + \frac{x\ (-1-y+y\ a+y\ b)\ A1}{y\ a\ x - x\ y - x - y\ b}$$

$$R = (B\ C1)\ \cap\ (B1\ C)$$

$$B + \frac{(-1-y+a+y\ a+b+y\ b)\ C1}{a\ x-b} = \frac{(-1-y+y\ a+y\ b)\ B1}{a\ x-b} + \frac{(1+x)\ a\ C}{a\ x-b}$$

$$R = \frac{y\ (-1+a+b)\ a\ A}{a\ x-1-y+a+y\ a+y\ b} + \frac{\left(-y\ b+b\ y\ a+y\ b^2+a\ x-b\right)\ B}{a\ x-1-y+a+y\ a+y\ b} - \frac{(-1+a+b)\ (-1-y+y\ a+y\ b)\ A1}{a\ x-1-y+a+y\ a+y\ b}$$

$$R1 = (B\ C1)\ \cap\ (P\ Q)$$

$$B + \frac{(-1-y+a+y\ a+b+y\ b)\ C1}{a\ x-b} = \frac{\left(1-a+y-2\ y\ a+a^2\ y-y\ b+b\ y\ a\right)\ P}{b} - \frac{(-1+a+b)\ a\ (y\ a\ x-x\ y-x-y\ b)\ Q}{b\ (a\ x-b)}$$

$$R1 = \frac{y\ (-1+a+b)\ a\ A}{a\ x-1-y+a+y\ a+y\ b} + \frac{\left(-y\ b+b\ y\ a+y\ b^2+a\ x-b\right)\ B}{a\ x-1-y+a+y\ a+y\ b} - \frac{(-1+a+b)\ (-1-y+y\ a+y\ b)\ A1}{a\ x-1-y+a+y\ a+y\ b}$$

$$R = R1$$

Statistics

Statistics for the proving time of 110 theorems in CH

Proving time (s)	Number of theorems	Percentage share of the total (%)
<0.1	88	80.0
<0.2	101	91.8
<0.5	109	99.1
<1.1	110	100

Our method is efficient!

Readability

The readability of the machine proofs produced by MPM-Prover is satisfactory:

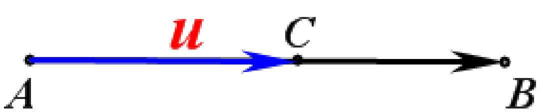
Clear and concise;
Simple and easy to understand;
With clear geometric meaning;
Rich information included.

Complex Coefficient Mass Point Geometry

1. *The basic idea of complex coeffcient mass point*
2. *Description of the linear constructive geometry statements*
3. *Algrithm CMPM*
4. *Implimentation of algorithm CMPM*
5. *Running examples*

Basic Idea of Complex Coefficient Mass Point

Real coefficient mass point geometry

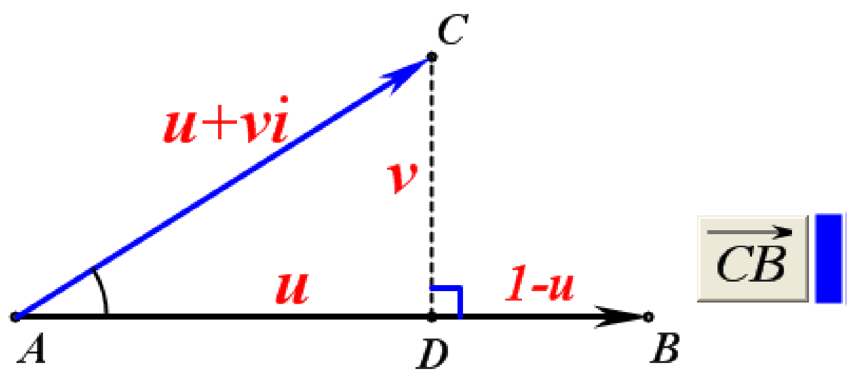


$$C - A = u\,(B - A)$$

$$\Rightarrow\quad C = (1-u)\,A + uB$$

$$\frac{\overrightarrow{AB}}{1} = \frac{\overrightarrow{AC}}{u} = \frac{\overrightarrow{CB}}{1-u}$$

Complex coefficient mass point geometry



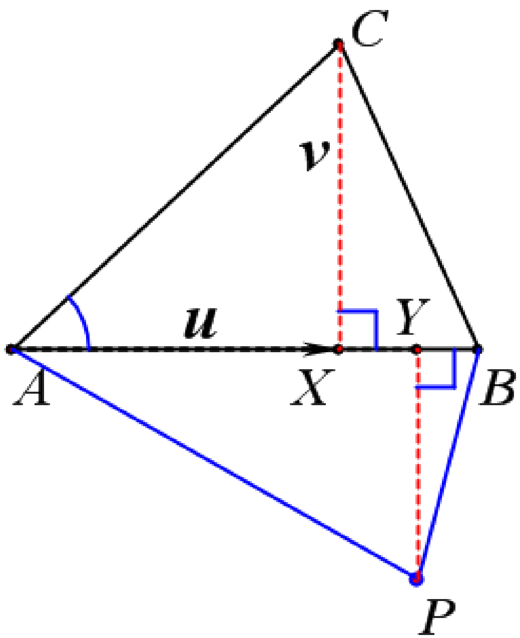
$$C - A = (u+vi)\,(B - A)$$

$$\Rightarrow\quad C = (1-u-vi)\,A + (u+vi)\,B$$

$$D = (1-u)\,A + uB\,,\quad C - D = vi\,(B - A)$$

$$\frac{\overrightarrow{AB}}{1} = \frac{\overrightarrow{AC}}{u+vi} = \frac{\overrightarrow{CB}}{1-u-vi}$$

Basic Properties of the Complex Coefficient Mass Point Geometry



$$C = (1-u-vi)A + (u+vi)B \; (u, v \in \mathrm{R})$$

When v>0, A-B-C is counterclockwise.
When v<0, A-B-C is clockwise.

$$(1)\; CX \perp AB \Rightarrow X = (1-u)A + uB$$

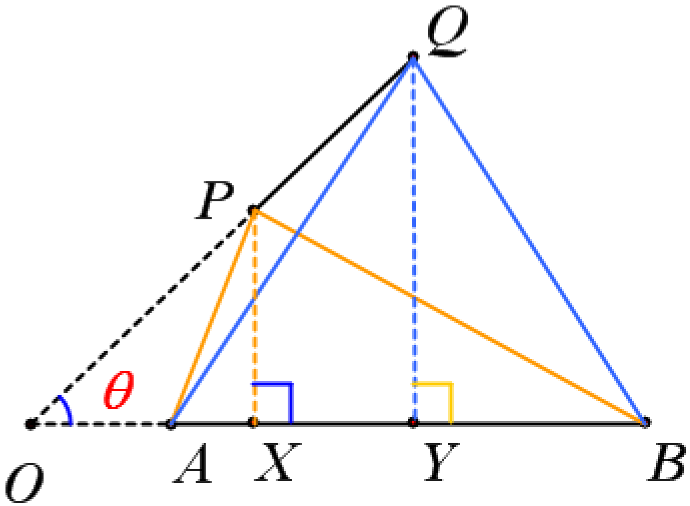
$$(2)\; u > 0 \Rightarrow \angle CAB < 90^\circ;$$

$$u = 0 \Rightarrow \angle CAB = 90^\circ;$$

$$u < 0 \Rightarrow \angle CAB > 90^\circ.$$

P

Any point P on plane ABC can be written as the linear combination of A,B uniquely.



$$P-Q=(x+yi)(B-A)$$

$$\Rightarrow \quad \tan\theta=\frac{y}{x},\ \frac{|PQ|}{|AB|}=|x+yi|$$

$\Rightarrow$

(1) If $y=0,\ x\neq 0$, then $PQ\,/\!/\,AB$;

(2) If $y\neq 0,\ x=0$, then $PQ\perp AB$;

(3) If $|x+yi|=t$, then $|PQ|=t\,|AB|$.

when $t=1$, $|PQ|=|AB|$.

x=0

P

$$P=(1-m-ni)A+(m+ni)B$$

Q

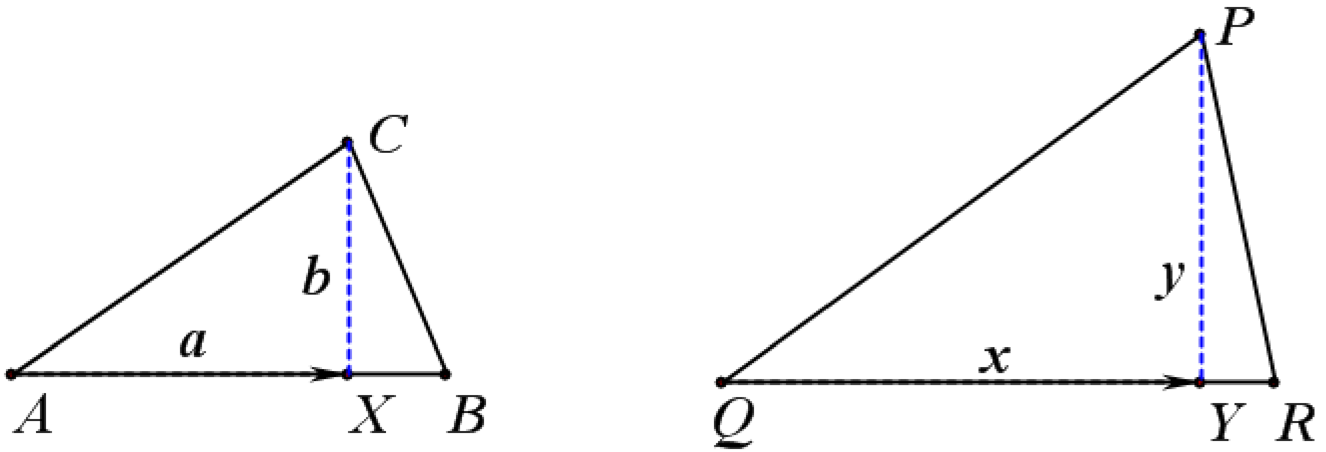
$$Q=(1-a-bi)A+(a+bi)B$$

$\Rightarrow$

(1) $\overline{PX} : \overline{QY} = n:b$;

(2) If $n\neq b$, $O = PQ\cap AB$, then

$(n\text{-}b)O = nQ - bP$.



$$C=(1-a-bi)A+(a+bi)B \qquad P=(1-x-yi)Q+(x+yi)R$$

(1) If $a=x=0$, then $\angle CAB=\angle PQR=90^\circ$;

(2) If $a\neq 0$, $x\neq 0$, and $\left|\dfrac{a}{b}\right|=\left|\dfrac{x}{y}\right|$,

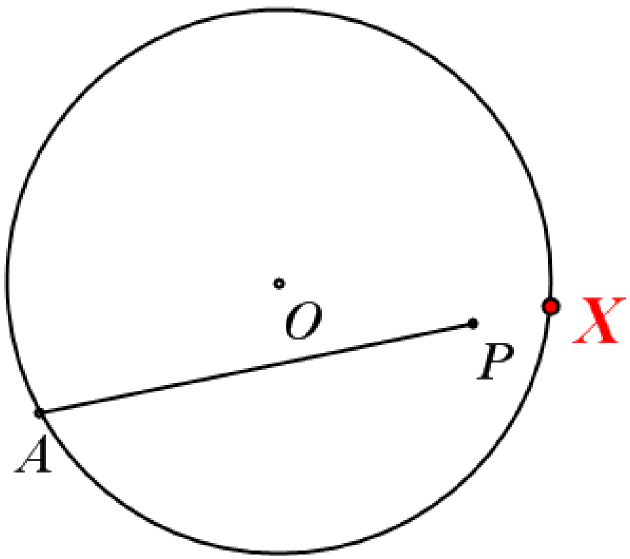
when $\dfrac{a}{x}=\left|\dfrac{b}{y}\right|$, then $\angle CAB=\angle PQR$;

when $\dfrac{a}{x}=-\left|\dfrac{b}{y}\right|$, then $\angle CAB+\angle PQR=180^\circ$;

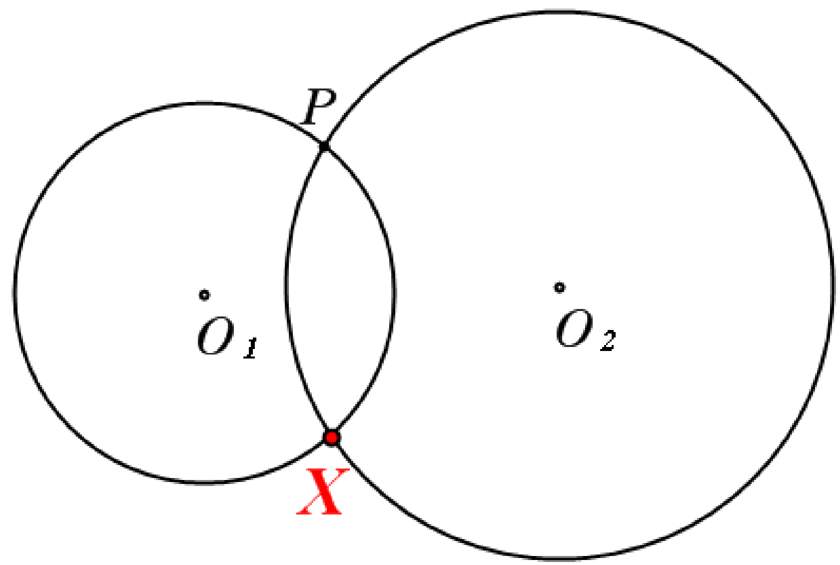
(3) If $a=x\neq 0, b=\pm y\neq 0$, then $\triangle CAB\backsim\triangle PQR$.

Linear Constructive Geometry Statements(CL)

The linear constructive geometry statements (CL) requests that the point introduced by each construction must be single.



$Cinter(X,O,A,P)$



$CCinter(X,P,O_1,O_2)$

Constructions in CL

Basic Constructions of CL

$Point(X)$

$Points(A,B)$

$Point1(X,A,B,a,b)$

$Lratio(X,A,B,r)$

$Mratio(X,A,B,r)$

$Midpoint1(X,A,B)$

$Symmetry(X,A,B)$

$Pratio(X,U,A,B,r)$

$Parallel(X,U,A,B)$

$Tratio(X,A,B,r)$

$Bratio(X,A,B,r)$

$Inter(X,U,V,A,B)$

$Pinter(X,A,B,W,U,V)$

$PPinter(X,C,A,B,W,U,V)$

$Tinter(X,A,B,W,U,V)$

$Foot1(X,C,A,B)$

$Binter(X,A,B,U,V)$

$BBinter(X,A,B,U,V)$

$PTinter(X,C,A,B,W,U,V)$

$Cinter(X,O,A,P)$

$CCinter(X,P,O_1,O_2)$

$Pointoncircle(X,O,A,\cos r,\sin r)$

$Similartrianglepoint(X,P,Q,A,B,C)$

$Similartrianglepoint1(X,P,Q,A,B,C)$

Combined Constructions

$Centroid(X,A,B,C)$

$Circumcenter(X,A,B,C)$

$Orthocenter(X,A,B,C)$

$Incenter(X,I,B,C)$

$Symmetricpoint(X,P,A,B)$

$Conjugatepoint(X,A,B,C,P)$

$Pointoncircle1(X,A,B,C,P)$

$ASApoint(X,A,B,\alpha,\beta)$

....

Check \ Evaluation Functions in CL

$relation3(P,Q,R)$	Find a,b such that P=(1-a-bi)Q+(a+bi)R;
$relation4(P,Q,M,N)$	Find x,y such that P-Q=(x+yi)(M-N);
$equalproduct(P,Q,M,N,U,V,S,T)$	Check whether $\|PQ\|*\|MN\| = \|UV\|*\|ST\|$;
$equalangle(P,Q,R,A,B,C)$	Check whether $\angle PQR = \angle CAB$;
$cocircle(P,Q,R,S)$	Check whether P,Q,R,S are concyclic;
$similartriangle(P,Q,R,A,B,C)$	Check whether $\triangle PQR \backsim \triangle CAB$;
$cctangent(O_1,P,O_2,Q)$	Check whether two circles are tangent or not;
$onradical(X,O_1,P,O_2,Q)$	Check whether X lies on the axis of the two circles;
$inversion(P,Q,O,A)$	Check whether P is the inversion of Q wrt the circle $\odot(O,A)$;
$area3(P,Q,R)$	Find the signed area of oriented triangle PQR;
$arearatio(P,Q,R,A,B,C)$	Find the ratio of two signed area of oriented triangles ABC an

Algorithm CMPM *(Complex-Mass-Point-Method)*

Algorithm CMPM is similar to algorithm MPM.

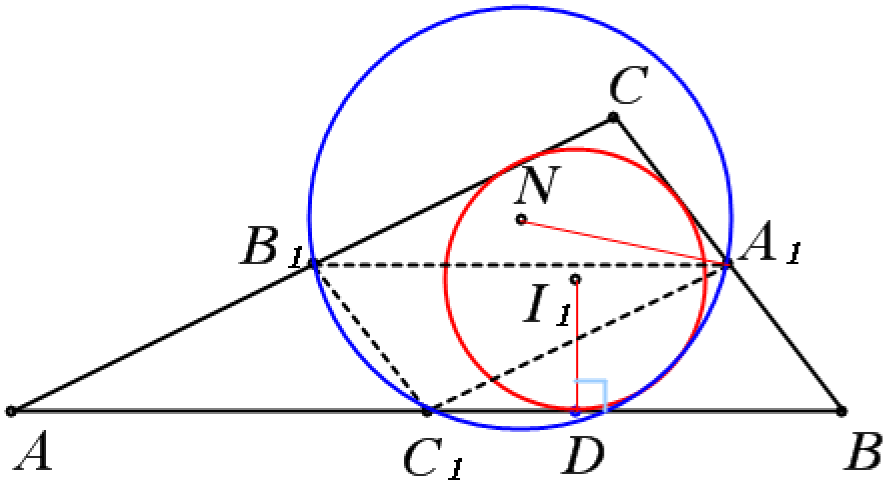
CMPM
- **CCF** — *construction function set*
- **CAF** — *objectives function set*
- **CMMC** — *controller*

A constructive statement (S) in class CL:

constructions $\left(C_1, C_2, \cdots, C_k\right)$ + *objectives* $\left(E_1^*, E_2^*, \cdots\right)$

$$S = \left(C_1, C_2, \cdots, C_k, G = \left(E_1^*, E_2^*, \cdots\right)\right)$$

Running example 3: Feuerbach's Theorem




```
Points(A, B);
Point1(I1, A, B, a, b);
Incenter(C, A, B, I1);
Midpoint1(A1, B, C);
Midpoint1(B1, A, C);
Midpoint1(C1, A, B);
Circumcenter(N, A1, B1, C1);
Foot1(D, I1, A, B);
CCtangent(N, A1, I1, D);
```

Machine Proof for Feuerbach's Theorem

$$A,\ B$$

$$I1 = (1 - a^{\sim} - I\ b^{\sim})\ A + (a^{\sim} + I\ b^{\sim})\ B$$

$$C = -\frac{a^{\sim}\left(a^{\sim 2} + 2\ I\ b^{\sim} a^{\sim} - 2\ a^{\sim} - 2\ I\ b^{\sim} + 1 - b^{\sim 2}\right) A}{-a^{\sim} + a^{\sim 2} + b^{\sim 2}} + \frac{\left(a^{\sim 3} - a^{\sim 2} - a^{\sim} b^{\sim 2} + b^{\sim 2} - 2\ I\ b^{\sim} a^{\sim} + 2\ I\ a^{\sim 2} b^{\sim}\right) B}{-a^{\sim} + a^{\sim 2} + b^{\sim 2}}$$

$$2\ A1 = B + C$$

$$2\ B1 = A + C$$

$$2\ C1 = A + B$$

$$N = \frac{1}{8}\ \frac{\left(-4\ b^{\sim} a^{\sim 3} - 6\ I\ a^{\sim 2} b^{\sim 2} + 6\ I\ b^{\sim 2} a^{\sim} - I\ b^{\sim 2} + I\ b^{\sim 4} - 6\ a^{\sim} b^{\sim} + 4\ a^{\sim} b^{\sim 3} - 2\ I\ a^{\sim 3} + I\ a^{\sim 4} + I\ a^{\sim 2} + 2\ b^{\sim 3} + 10\ b^{\sim} a^{\sim 2}\right) A}{b^{\sim}\left(-a^{\sim} + a^{\sim 2} + b^{\sim 2}\right)}$$
$$-\frac{1}{8}\ \frac{\left(-4\ b^{\sim} a^{\sim 3} - 6\ I\ a^{\sim 2} b^{\sim 2} + 6\ I\ b^{\sim 2} a^{\sim} - I\ b^{\sim 2} + I\ b^{\sim 4} + 2\ a^{\sim} b^{\sim} + 4\ a^{\sim} b^{\sim 3} - 2\ I\ a^{\sim 3} + I\ a^{\sim 4} + I\ a^{\sim 2} - 6\ b^{\sim 3} + 2\ b^{\sim} a^{\sim 2}\right) B}{b^{\sim}\left(-a^{\sim} + a^{\sim 2} + b^{\sim 2}\right)}$$

$$(I1\ D)\ \perp\ (B\ A)$$

$$I1 = (1 - a^{\sim} - I\ b^{\sim})\ A + (a^{\sim} + I\ b^{\sim})\ B$$

$$D = (1 - a^{\sim})\ A + a^{\sim} B$$

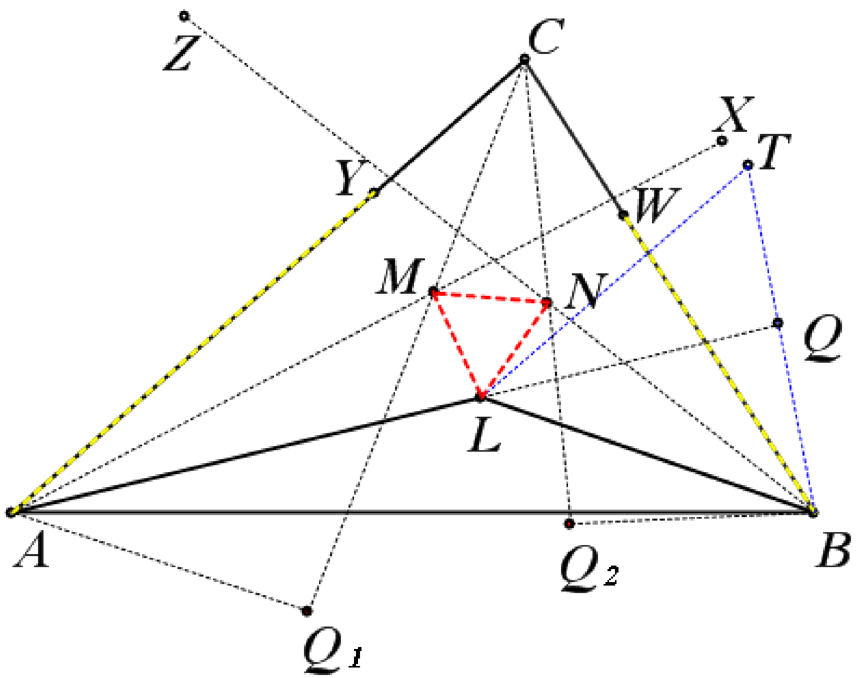
$$N - A1 = -\frac{1}{8}\ \frac{\left(2\ b^{\sim 3} - 2\ a^{\sim} b^{\sim} + 2\ b^{\sim} a^{\sim 2} + 2\ I\ a^{\sim 2} b^{\sim 2} - 2\ I\ b^{\sim 2} a^{\sim} - I\ b^{\sim 2} + I\ b^{\sim 4} - 2\ I\ a^{\sim 3} + I\ a^{\sim 4} + I\ a^{\sim 2}\right)(B - A)}{b^{\sim}\left(-a^{\sim} + a^{\sim 2} + b^{\sim 2}\right)}$$

$$I1 - D = I\ b^{\sim}\ (B - A)$$

$$N - I1 = -\frac{1}{8}\ \frac{\left(4\ b^{\sim} a^{\sim 3} + 12\ a^{\sim} b^{\sim 3} - 6\ b^{\sim 3} + 2\ a^{\sim} b^{\sim} - 6\ b^{\sim} a^{\sim 2} + 2\ I\ a^{\sim 2} b^{\sim 2} - 2\ I\ b^{\sim 2} a^{\sim} - I\ b^{\sim 2} + 9\ I\ b^{\sim 4} - 2\ I\ a^{\sim 3} + I\ a^{\sim 4} + I\ a^{\sim 2}\right)(B - A)}{b^{\sim}\left(-a^{\sim} + a^{\sim 2} + b^{\sim 2}\right)}$$

"They are tangent!"

Running example 4: Morley's Theorem



$$\Rightarrow \quad L-N=\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\,i\right)(M-N)$$

```
Points(A, B);
Point1(L, A, B, a, b);
Symmetricpoint(X, B, A, L);
Symmetricpoint(Y, L, A, X);
Symmetricpoint(Z, A, B, L);
Symmetricpoint(W, L, B, Z);
Inter(C, A, Y, B, W);
Bratio(T, L, B, √3/2);
Inter(Q, A, L, B, T);
Similartrianglepoint1(Q1, C, A, Q, L, T);
Similartrianglepoint(Q2, C, B, Q, L, T);
Inter(M, A, X, C, Q1);
Inter(N, B, Z, C, Q2);
relation4jj(M, N, L, N);
```

$$A,\ B$$

$$L=(1-a^{\sim}-I\ b^{\sim})\ A+(a^{\sim}+I\ b^{\sim})\ B$$

$$B=\frac{\left(b^{\sim 2}-a^{\sim}+a^{\sim 2}+I\ b^{\sim}\right)\ A}{a^{\sim 2}+b^{\sim 2}}-\frac{(-a^{\sim}+I\ b^{\sim})\ L}{a^{\sim 2}+b^{\sim 2}}$$

$$X=-\frac{\left(a^{\sim}-a^{\sim 2}-b^{\sim 2}+I\ b^{\sim}\right)\ A}{a^{\sim 2}+b^{\sim 2}}+\frac{(a^{\sim}+I\ b^{\sim})\ L}{a^{\sim 2}+b^{\sim 2}}$$

$$L=(-a^{\sim}+I\ b^{\sim}+1)\ A+(a^{\sim}-I\ b^{\sim})\ X$$

$$Y=(1-a^{\sim}-I\ b^{\sim})\ A+(a^{\sim}+I\ b^{\sim})\ X$$

$$A=-\frac{\left(a^{\sim}-a^{\sim 2}-b^{\sim 2}+I\ b^{\sim}\right)\ B}{1-2\ a^{\sim}+a^{\sim 2}+b^{\sim 2}}+\frac{(-a^{\sim}+I\ b^{\sim}+1)\ L}{1-2\ a^{\sim}+a^{\sim 2}+b^{\sim 2}}$$

$$Z=\frac{\left(b^{\sim 2}-a^{\sim}+a^{\sim 2}+I\ b^{\sim}\right)\ B}{1-2\ a^{\sim}+a^{\sim 2}+b^{\sim 2}}-\frac{(-1+a^{\sim}+I\ b^{\sim})\ L}{1-2\ a^{\sim}+a^{\sim 2}+b^{\sim 2}}$$

$$L=(a^{\sim}-I\ b^{\sim})\ B+(-a^{\sim}+I\ b^{\sim}+1)\ Z$$

$$W=(a^{\sim}+I\ b^{\sim})\ B+(1-a^{\sim}-I\ b^{\sim})\ Z$$

$$Q=-\frac{1}{2}\Big(\big(-6\ I\ \sqrt{3}\ a^{\sim 3}+3\ a^{\sim 4}-6\ a^{\sim 3}+6\ I\ \sqrt{3}\ a^{\sim 2}\ b^{\sim 2}+3\ I\ b^{\sim 4}\ \sqrt{3}+6\ a^{\sim 2}\ b^{\sim 2}+3\ a^{\sim 2}+3\ I\ \sqrt{3}\ a^{\sim 4}-6\ a^{\sim}\ b^{\sim 2}-6\ I\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}-b^{\sim 2}-I\ \sqrt{3}\ b^{\sim 2}+3\ b^{\sim 4}+3\ I\ \sqrt{3}\ a^{\sim 2}\big)\ L\Big)\Big/\big(-3\ a^{\sim 4}+6\ a^{\sim 3}+2\ b^{\sim}\ \sqrt{3}\ a^{\sim 2}-6\ a^{\sim 2}\ b^{\sim 2}-3\ a^{\sim 2}+6\ a^{\sim}\ b^{\sim 2}-2\ b^{\sim}\ \sqrt{3}\ a^{\sim}-b^{\sim 2}+2\ b^{\sim 3}\ \sqrt{3}-3\ b^{\sim 4}\big)-\frac{1}{2}\Big(\big(\sqrt{3}\ a^{\sim 4}-2\ \sqrt{3}\ a^{\sim 3}+2\ \sqrt{3}\ a^{\sim 2}\ b^{\sim 2}+\sqrt{3}\ a^{\sim 2}-4\ b^{\sim}\ a^{\sim 2}+4\ a^{\sim}\ b^{\sim}-2\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}+b^{\sim 4}\ \sqrt{3}+\sqrt{3}\ b^{\sim 2}-4\ b^{\sim 3}-3\ I\ a^{\sim 4}-6\ I\ a^{\sim 2}\ b^{\sim 2}+6\ I\ a^{\sim 3}+I\ b^{\sim 2}+6\ I\ b^{\sim 2}\ a^{\sim}-3\ I\ b^{\sim 4}-3\ I\ a^{\sim 2}\big)\ \sqrt{3}\ T\Big)\Big/\big(-3\ a^{\sim 4}+6\ a^{\sim 3}+2\ b^{\sim}\ \sqrt{3}\ a^{\sim 2}-6\ a^{\sim 2}\ b^{\sim 2}-3\ a^{\sim 2}+6\ a^{\sim}\ b^{\sim 2}-2\ b^{\sim}\ \sqrt{3}\ a^{\sim}-b^{\sim 2}+2\ b^{\sim 3}\ \sqrt{3}-3\ b^{\sim 4}\big)$$

$$Q2=-\frac{1}{2}\Big(\big(-6\ I\ \sqrt{3}\ a^{\sim 3}+3\ a^{\sim 4}-6\ a^{\sim 3}+6\ I\ \sqrt{3}\ a^{\sim 2}\ b^{\sim 2}+3\ I\ b^{\sim 4}\ \sqrt{3}+6\ a^{\sim 2}\ b^{\sim 2}+3\ a^{\sim 2}+3\ I\ \sqrt{3}\ a^{\sim 4}-6\ a^{\sim}\ b^{\sim 2}-6\ I\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}-b^{\sim 2}-I\ \sqrt{3}\ b^{\sim 2}+3\ b^{\sim 4}+3\ I\ \sqrt{3}\ a^{\sim 2}\big)\ C\Big)\Big/\big(-3\ a^{\sim 4}+6\ a^{\sim 3}+2\ b^{\sim}\ \sqrt{3}\ a^{\sim 2}-6\ a^{\sim 2}\ b^{\sim 2}-3\ a^{\sim 2}+6\ a^{\sim}\ b^{\sim 2}-2\ b^{\sim}\ \sqrt{3}\ a^{\sim}-b^{\sim 2}+2\ b^{\sim 3}\ \sqrt{3}-3\ b^{\sim 4}\big)-\frac{1}{2}\Big(\big(\sqrt{3}\ a^{\sim 4}-2\ \sqrt{3}\ a^{\sim 3}+2\ \sqrt{3}\ a^{\sim 2}\ b^{\sim 2}+\sqrt{3}\ a^{\sim 2}-4\ b^{\sim}\ a^{\sim 2}+4\ a^{\sim}\ b^{\sim}-2\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}+b^{\sim 4}\ \sqrt{3}+\sqrt{3}\ b^{\sim 2}-4\ b^{\sim 3}-3\ I\ a^{\sim 4}-6\ I\ a^{\sim 2}\ b^{\sim 2}+6\ I\ a^{\sim 3}+I\ b^{\sim 2}+6\ I\ b^{\sim 2}\ a^{\sim}-3\ I\ b^{\sim 4}-3\ I\ a^{\sim 2}\big)\ \sqrt{3}\ B\Big)\Big/\big(-3\ a^{\sim 4}+6\ a^{\sim 3}+2\ b^{\sim}\ \sqrt{3}\ a^{\sim 2}-6\ a^{\sim 2}\ b^{\sim 2}-3\ a^{\sim 2}+6\ a^{\sim}\ b^{\sim 2}-2\ b^{\sim}\ \sqrt{3}\ a^{\sim}-b^{\sim 2}+2\ b^{\sim 3}\ \sqrt{3}-3\ b^{\sim 4}\big)$$

$$M=(A\ X)\ \cap\ (C\ Q1)$$

$$\frac{1}{\left(\sqrt{3}\ a^{\sim}-b^{\sim}\right)\ \left(6\ a^{\sim 2}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim 2}+3\ b^{\sim 4}+3\ a^{\sim 2}-6\ a^{\sim 3}+3\ a^{\sim 4}-b^{\sim 2}\right)}\Big(\big(6\ \sqrt{3}\ a^{\sim 3}\ b^{\sim 2}-10\ \sqrt{3}\ a^{\sim 2}\ b^{\sim 2}+3\ \sqrt{3}\ a^{\sim}\ b^{\sim 4}+3\ \sqrt{3}\ a^{\sim 3}-6\ \sqrt{3}\ a^{\sim 4}+3\ \sqrt{3}\ a^{\sim 5}+3\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}+6\ b^{\sim 3}\ a^{\sim 2}-6\ a^{\sim}\ b^{\sim 3}+3\ b^{\sim 5}+3\ b^{\sim}\ a^{\sim 2}-6\ b^{\sim}\ a^{\sim 3}+3\ b^{\sim}\ a^{\sim 4}+3\ b^{\sim 3}-4\ b^{\sim 4}\ \sqrt{3}\big)\ M\Big)=C-\frac{2\ \left(-3\ a^{\sim 4}+6\ a^{\sim 3}+2\ b^{\sim}\ \sqrt{3}\ a^{\sim 2}-6\ a^{\sim 2}\ b^{\sim 2}-3\ a^{\sim 2}+6\ a^{\sim}\ b^{\sim 2}-2\ b^{\sim}\ \sqrt{3}\ a^{\sim}-b^{\sim 2}+2\ b^{\sim 3}\ \sqrt{3}-3\ b^{\sim 4}\right)\ b^{\sim}\ Q1}{\left(\sqrt{3}\ a^{\sim}-b^{\sim}\right)\ \left(6\ a^{\sim 2}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim 2}+3\ b^{\sim 4}+3\ a^{\sim 2}-6\ a^{\sim 3}+3\ a^{\sim 4}-b^{\sim 2}\right)}$$

$$M=-\Big(\big(3\ \sqrt{3}\ a^{\sim 4}-9\ \sqrt{3}\ a^{\sim 3}+6\ I\ \sqrt{3}\ a^{\sim 3}\ b^{\sim}-2\ I\ \sqrt{3}\ a^{\sim}\ b^{\sim 3}-4\ \sqrt{3}\ a^{\sim 2}\ b^{\sim 2}+9\ \sqrt{3}\ a^{\sim 2}-3\ b^{\sim}\ a^{\sim 2}-12\ I\ \sqrt{3}\ a^{\sim 2}\ b^{\sim}+6\ a^{\sim}\ b^{\sim}+3\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}+6\ I\ \sqrt{3}\ a^{\sim}\ b^{\sim}-3\ \sqrt{3}\ a^{\sim}-3\ b^{\sim}+\sqrt{3}\ b^{\sim 2}-3\ b^{\sim 3}+b^{\sim 4}\ \sqrt{3}\big)\ A\Big)\Big/\big(3\ \sqrt{3}\ a^{\sim 3}-6\ \sqrt{3}\ a^{\sim 2}+3\ b^{\sim}\ a^{\sim 2}+3\ \sqrt{3}\ a^{\sim}+3\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim}+3\ b^{\sim 3}+3\ b^{\sim}-4\ \sqrt{3}\ b^{\sim 2}\big)+\frac{\left(3\ a^{\sim 4}-6\ a^{\sim 3}+6\ I\ b^{\sim}\ a^{\sim 3}-12\ I\ a^{\sim 2}\ b^{\sim}-4\ a^{\sim 2}\ b^{\sim 2}+3\ a^{\sim 2}-2\ I\ a^{\sim}\ b^{\sim 3}+6\ I\ b^{\sim}\ a^{\sim}+6\ a^{\sim}\ b^{\sim 2}-3\ b^{\sim 2}+b^{\sim 4}\right)\ \sqrt{3}\ B}{3\ \sqrt{3}\ a^{\sim 3}-6\ \sqrt{3}\ a^{\sim 2}+3\ b^{\sim}\ a^{\sim 2}+3\ \sqrt{3}\ a^{\sim}+3\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim}+3\ b^{\sim 3}+3\ b^{\sim}-4\ \sqrt{3}\ b^{\sim 2}}$$

$$C=(A\ Y)\ \cap\ (B\ W)$$

$$\frac{1}{4}\ \frac{\left(6\ a^{\sim 2}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim 2}+3\ b^{\sim 4}+3\ a^{\sim 2}-6\ a^{\sim 3}+3\ a^{\sim 4}-b^{\sim 2}\right)\ C}{b^{\sim 2}\ \left(a^{\sim 2}-2\ a^{\sim}+b^{\sim 2}\right)}=B+\frac{1}{4}\ \frac{\left(1-2\ a^{\sim}+a^{\sim 2}+\right.}{b^{\sim 2}\ \left(a^{\sim 2}\right.}$$

$$C=-\frac{\left(-3\ a^{\sim 2}+3\ a^{\sim}-1+3\ b^{\sim 2}+a^{\sim 3}-3\ a^{\sim}\ b^{\sim 2}+3\ I\ a^{\sim 2}\ b^{\sim}-6\ I\ a^{\sim}\ b^{\sim}+3\ I\ b^{\sim}-I\ b^{\sim 3}\right)\ \left(3\ a^{\sim 2}-b^{\sim 2}\right)\ A}{6\ a^{\sim 2}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim 2}+3\ b^{\sim 4}+3\ a^{\sim 2}-6\ a^{\sim 3}+3\ a^{\sim 4}-b^{\sim 2}}+\frac{1}{6\ a^{\sim 2}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim 2}+3\ b^{\sim 4}+3\ a^{\sim 2}-6\ a^{\sim 3}+3\ a^{\sim 4}-b^{\sim 2}}\Big(\big(18\ a^{\sim 2}\ b^{\sim 2}-9\ a^{\sim}\ b^{\sim 2}-6\ a^{\sim 4}+3\ a^{\sim 3}+3\ a^{\sim 5}+9\ I\ a^{\sim 2}\ b^{\sim}-6\ I\ b^{\sim 3}\ a^{\sim 2}+3\ a^{\sim}\ b^{\sim 4}+6\ I\ a^{\sim}\ b^{\sim 3}-3\ I\ b^{\sim 3}+I\ b^{\sim 5}\big)\ B\Big)$$

$$T=\left(\frac{1}{2}-\frac{1}{2}\ I\ \sqrt{3}\right)\ L+\left(\frac{1}{2}\ I\ \sqrt{3}+\frac{1}{2}\right)\ B$$

$$Q=(A\ L)\ \cap\ (B\ T)$$

$$\left(-\sqrt{3}\ a^{\sim}+\sqrt{3}\ a^{\sim 2}-b^{\sim}+\sqrt{3}\ b^{\sim 2}\right)\ Q\ \_ \qquad 2\ b^{\sim}\ T$$

$$N=(B\ Z)\ \cap\ (C\ Q2)$$

$$\Big(\big(-3\ b^{\sim 3}+3\ \sqrt{3}\ a^{\sim 5}-9\ \sqrt{3}\ a^{\sim 4}+9\ \sqrt{3}\ a^{\sim 3}+b^{\sim 4}\ \sqrt{3}+\sqrt{3}\ b^{\sim 2}-3\ b^{\sim}\ a^{\sim 2}-3\ b^{\sim 5}-6\ b^{\sim 3}\ a^{\sim 2}-3\ b^{\sim}\ a^{\sim 4}+a^{\sim}\ \sqrt{3}\ b^{\sim 2}+3\ \sqrt{3}\ a^{\sim}\ b^{\sim 4}-8\ \sqrt{3}\ a^{\sim 2}+6\ \sqrt{3}\ a^{\sim 3}\ b^{\sim 2}-3\ \sqrt{3}\ a^{\sim 2}+6\ b^{\sim}\ a^{\sim 3}+6\ a^{\sim}\ b^{\sim 3}\big)\ N\Big)\Big/\Big(\left(\sqrt{3}\ a^{\sim}+b^{\sim}-\sqrt{3}\right)\ \left(6\ a^{\sim 2}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim 2}+3\ b^{\sim 4}+3\ a^{\sim 2}-6\ a^{\sim 3}+3\ a^{\sim 4}-b^{\sim 2}\right)\Big)=C+\frac{2\ \left(-3\ a^{\sim 4}+6\ a^{\sim 3}+2\ b^{\sim}\ \sqrt{3}\ a^{\sim 2}-6\ a^{\sim 2}\ b^{\sim 2}-3\ a^{\sim 2}+6\ a^{\sim}\ b^{\sim 2}-2\ b^{\sim}\ \sqrt{3}\ a^{\sim}-b^{\sim 2}+2\ b^{\sim 3}\ \sqrt{3}-3\ b^{\sim 4}\right)\ b^{\sim}\ Q2}{\left(\sqrt{3}\ a^{\sim}+b^{\sim}-\sqrt{3}\right)\ \left(6\ a^{\sim 2}\ b^{\sim 2}-6\ a^{\sim}\ b^{\sim 2}+3\ b^{\sim 4}+3\ a^{\sim 2}-6\ a^{\sim 3}+3\ a^{\sim 4}-b^{\sim 2}\right)}$$

$$N=-\frac{\left(a^{\sim 2}+2\ I\ b^{\sim}\ a^{\sim}-2\ a^{\sim}-2\ I\ b^{\sim}+1-b^{\sim 2}\right)\ \sqrt{3}\ \left(3\ a^{\sim 2}-b^{\sim 2}\right)\ A}{3\ \sqrt{3}\ a^{\sim 3}-3\ b^{\sim}\ a^{\sim 2}-3\ \sqrt{3}\ a^{\sim 2}+3\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}-3\ b^{\sim 3}+\sqrt{3}\ b^{\sim 2}}+\frac{1}{3\ \sqrt{3}\ a^{\sim 3}-3\ b^{\sim}\ a^{\sim 2}-3\ \sqrt{3}\ a^{\sim 2}+3\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}-3\ b^{\sim 3}+\sqrt{3}\ b^{\sim 2}}\Big(\big(3\ \sqrt{3}\ a^{\sim 4}-3\ \sqrt{3}\ a^{\sim 3}+6\ I\ \sqrt{3}\ a^{\sim 3}\ b^{\sim}-3\ b^{\sim}\ a^{\sim 2}-4\ \sqrt{3}\ a^{\sim 2}\ b^{\sim 2}-6\ I\ \sqrt{3}\ a^{\sim 2}\ b^{\sim}-2\ I\ \sqrt{3}\ a^{\sim}\ b^{\sim 3}+5\ a^{\sim}\ \sqrt{3}\ b^{\sim 2}+b^{\sim 4}\ \sqrt{3}-3\ b^{\sim 3}+2\ I\ \sqrt{3}\ b^{\sim 3}\big)\ B\Big)$$

$$L-N=\left(\frac{1}{2}\ I\ \sqrt{3}+\frac{1}{2}\right)\ (M-N)$$

$$L\ N^{2}=M\ N^{2}$$

Proving Time

Proving Time of Examples

The name of theorems	Proving Time (s)
The Napoleon Triangle	0.436
A theorem on Orthocenter	0.187
Four theorems about Orthocenter	0.514
Feuerbach's Theorem	0.438
Morley's Triangle Theorem	1.778
Miquel's Theorem	1.575
Pascal's Theorem On a Circle	1.544
Theorem of Pratt-Wu	0.899
The General Butterfly Theorem for Circle	0.826
The Special Case of Five Circles Theorem	1.029

Statistics

Statistics for proving time of 240 theorems in class CL

Proving time (s)	Number of theorems	Percentage share of the total (%)
<0.1	49	20.4
<0.2	103	42.9
<0.3	144	60.0
<0.5	200	83.3
<1.0	227	94.6
<2.0	236	98.3
<20.0	240	100

Comparison: MPM and CMPM

MPM $\xrightarrow{\textit{designed only for}}$ **Statements in CH**

CH $\subset$ **CL**

CMPM $\xrightarrow{\textit{designed for}}$ **Statements in CL**

Comparison: *elimating point methods*

Mass-Point Method ***MPM\CMPM***

Dealing with geometric point directly;
Easy to be extended or combined;
A complete algorithm.

Other readable machine proving methods

the Area Method
the Vector Method
the Full-angle Method
the Geometric Algebra Method
the Advanced Invariants Method

Dealing with geometry problems using the geometric quantities;
Not so easy to be extended or combined;
Only the Area Method is a complete algorithm.

Conclusion

Develop a new readable machine proving method based on geometric points——Mass-Point Method;

Establish two machine proving algorithms MPM\CMPM;

Algorithms and programs for MPM\ CMPM are much more concise than that of the Area Method\ Vector Method.

The run results of hundreds of non-trivial statements show that our method is not only efficient, most proofs' readability are also satisfactory.

Thanks!

If you have any questions, please contact:

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