Formal Representation and Automated Transformation of Geometric Statements

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Outline

Start with Geometry Software

Geometry problems (drawing or proving) are specified by applying similar (or same) concepts which are implemented differently in these systems.

Table: Constructive style

Cinderella	GeoGebra
Perpendicular(a;A)	PerpendicularLine[A,a]
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The constructions and predicates can be viewed as concepts.

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• Intergeo project offers a common file format for specifying dynamic diagrams. However, the format only works for constructive style.

• GeoCode is a generic proof scheme standard providing routine codes that can be interfaced with different CAS or provers for proving and DGS for drawing.

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- GEOTHER provides a standard form for specifying the entries contained in the predicates routines. However, defined predicates are independent with each other.
- GeoCode provides the facility for users to define new functions in terms of exited functions. However, these functions are defined only in the constructive style.

Objectives

- A general geometry programming language is needed in which one can easily and naturally define geometric concepts and specify problems in terms of the customized concepts (for both constructive and constraint type).

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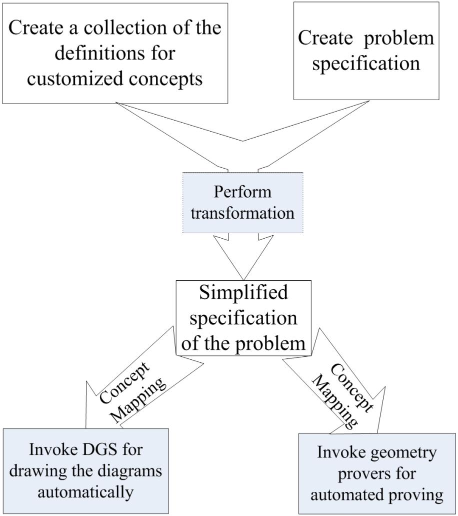
- A general geometry programming language is needed in which one can easily and naturally define geometric concepts and specify problems in terms of the customized concepts (for both constructive and constraint type).
- The facility is needed for transforming the specified problems into the ones that target systems can identify and manipulate via specific interfaces.

Idea



Create a collection of the definitions for customized concepts
Create problem specification
Perform transformation
Simplified specification of the problem
Concept Mapping
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Invoke DGS for drawing the diagrams automatically
Invoke geometry provers for automated proving

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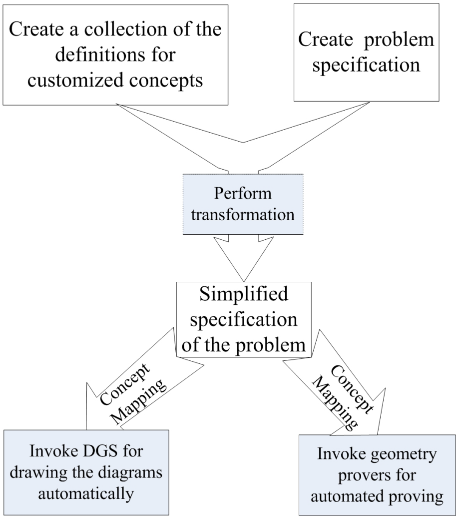

see Demo

Outline

Concept Symbols

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- **Built-in concepts**:
  - **Constants**: 0, π, etc.
  - **Pointers (labels)**: A, B, l, etc.
  - **Types**: Point, Line, Segment, Length, Degree, Number, Boolean, etc.
  - **Algebra concepts**: $times$, $plus$, sin, $squre$, etc.
  - **Set concepts**: $list$, $choose$, $ismember$, etc.
  - **Logic concepts**: and, or, not.

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- **Entity concepts**:
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  - **Geometric quantities**: length(segment(A::Point,B::Point)), ratio(a::GeometricQuantity,b::GeometricQuantity), etc.

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- **Boolean concepts**:
  - **Geometric relations**: parallel(l::Line,m::Line), isin(A::Point,o::Circle), tangent(o::Circle, p::Circle) etc.
  - **Quantity relations**: lt(a::Length,b::Length), equal(c::Degree,d::Degree), etc.

Constructing Geometric Clauses

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- **Reference clauses**: A:=point(), P:=intersection(l,m), etc.
- **Boolean clauses**: perpendicular(l,m), incident(A,l), etc.
- **Compound clauses**:
  - **Nesting**: collinear(foot(D,line(A,B)),foot(D,line(A,C)),foot(D,line(B,C)));
  - **Give**: give(triangle(A,B,C));
  - **Configuration**: configuration(E:=intersection(line(A,B),line(C,D)), F:=intersection(line(A,C),line(B,D)));
  - **Declare**: declare(A::Point,B::Point,l::Line);
  - **Logic**: and(parallel(l,m),incident(A,l));
  - **List**: {A;B;C},{point();point();midpoint(segment(A,B))};
  - **Set**: choosediff(A;B;C,2));
  - **Algebra**: times(2,length(segment(A,B))).

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- Definition(intersection(l::Line,m::Line), **[**A::Point **where** and(incident(A,l), incident(A,m))**]**, intersect(l,m))
- Definition(completequadrilateral(A::Point,B::Point,C::Point,D::Point, E::Point,F::Point), **[**configuration(E:=intersection(line(A,B),line(C,D)), F:=intersection(line(A,C),line(B,D)))**]**, null)

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- Definition(diagonal(completequadrilateral(A::Point,B::Point,C::Point,D::Point, E::Point,F::Point)),{**[**segment(A,D)**]**;**[**segment(B,C)**]**; **[**segment(E,F)**]**}, null)

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- Problem(Simson,Theorem,assume(A:=point(),B:=point(),C:=point(), D:=point(),incident(D,circumcircle(triangle(A,B,C)))), show(collinear(foot(D,line(A,B)),foot(D,line(A,C)),foot(D,line(B,C)))))

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- Problem(Pappus,Theorem,assume(declare(C::Point,F::Point,P::Point, Q::Point,R::Point),A:=point(),B:=point(),D:=point(),E:=point(), give(Pappus(A,B,C,D,E,F,P,Q,R))), show(collinear(P,Q,R)))

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[intersection(perpendicularline(D,line(E,F)),line(E,F))]

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We adopt eager (inner most) strategy to deal with the nesting cases.

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$$\xrightarrow[\textit{simplification}]{\textit{definitions}}$$

Problem(Simson,Theorem,assume(declare(var_0::Point,var_1::Point,var_2::Line,
var_3::Point,var_4::Line,var_5::Point,var_6::Line,var_7::Point),
A:=point(),B:=point(),C:=point(),D:=point(),equal(distance(var_0,D),distance
(var_0,var_1)),equal(distance(var_0,var_1),distance(var_0,A)),equal(distance
(var_0,A),distance(var_0,B)),equal(distance(var_0,A),distance(var_0,C),...),
show(incident(var_3,line(var_5,var_7))))

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Generally, type for instance is not equal to type for concept. How to match them?

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Type Matching Rule

Let I and C be an instance and a concept, if $\mathtt{Type}(I) \preceq \mathtt{Type}(C)$, then the definition of C can be used to simplify instance I.

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[I **where** constraint **context** configuration **with** nondegeneracyCondition]

The simplified instances will be normalized into this form at each step of simplification process.

Analysis

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Termination

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Usability

The simplified problem specifications can be interfaced with Geometry software systems.

More Demo

- Reuse definitions and problem specifications.

  ▸ Pappus,completeQuandrilateral,197,198

More Demo

- Reuse definitions and problem specifications.
  - Pappus,completeQuandrilateral,197,198
- Dealing with multiple returns.
  - example90, 180

Outline

Tools and Open Source Packages

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- GeoGebra;
- GEOTHER.

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The specifications are

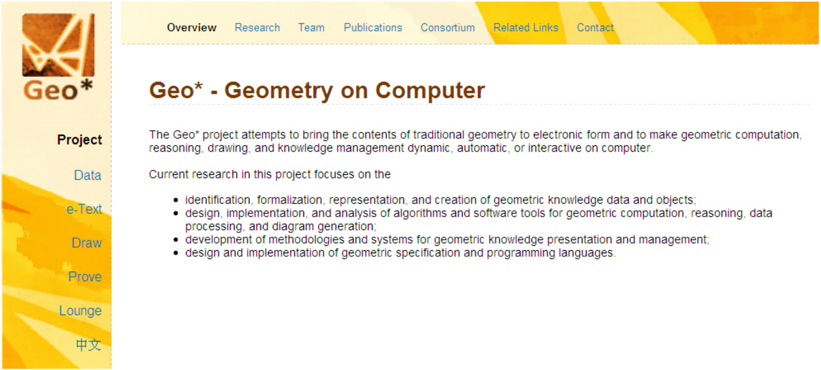
- encoded easily and naturally;
- used in both constraint and constructive cases;
- transformed into ones that can be interfaced with available geometry software systems.

Future Work

The geometry programming language is still at a preliminary stage. The following problems should be considered further.

- prove the correctness of transformation;
- transform the specifications in this language into natural language and the other way round;
- transform the specifications in this language into algebraic counterparts and interface with CAS.

Geo* - Geometry on Computer



Welcome to visit our project home at http://geo.cc4cm.org/

Thanks!