On the automatic discovery of Steiner-Lehmus generalizations

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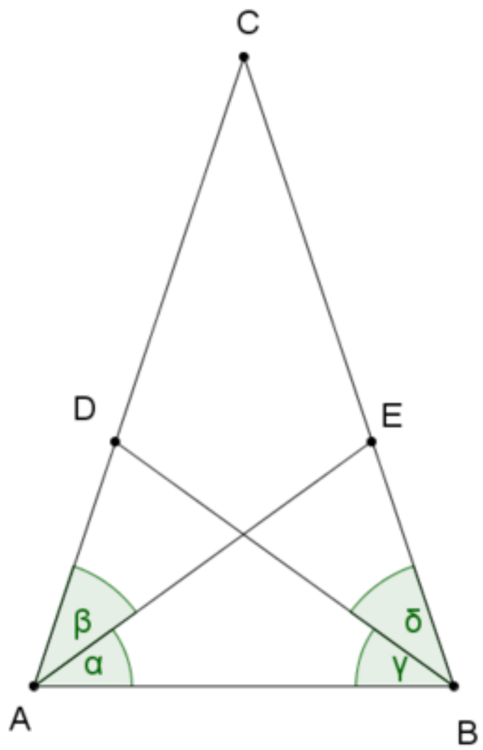
The Steiner-Lehmus Theorem (circa 1840)

- If a triangle has two (internal) angle-bisectors with the same length, i.e. if $|AE| = |BD|, \alpha = \beta, \gamma = \delta$



then the triangle must be isosceles

- The converse is, obviously, also true

The Steiner-Lehmus Theorem

- The theorem was first mentioned in 1840 in a letter by C. L. Lehmus to C. Sturm. Jakob Steiner was among the first to provide a solution.

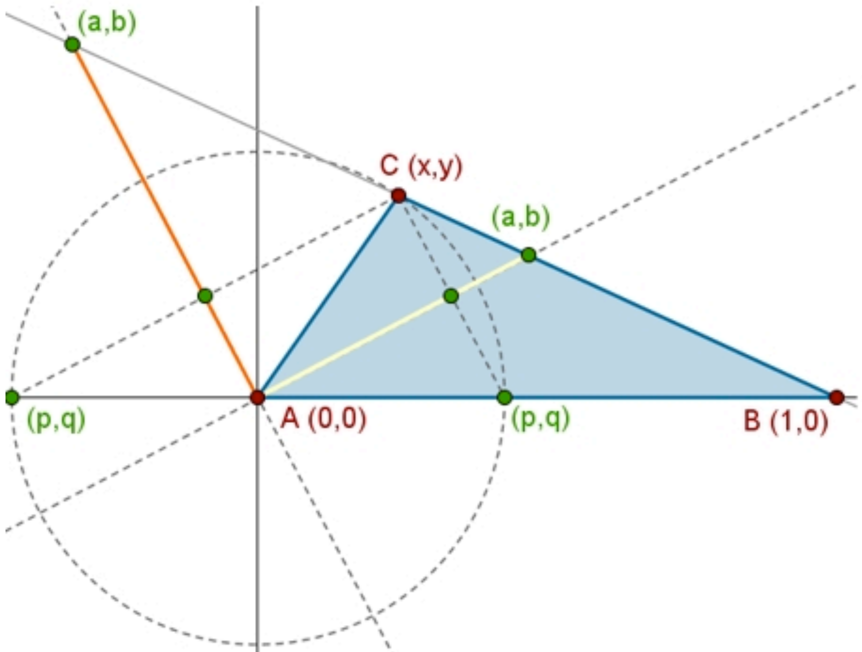
- The theorem became a rather popular topic in elementary geometry ever since, because of the difficulty to obtain a direct proof (P. Baptist, J. Conway, O. Bottema, V. Thebault…..)

See references at:

www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/steiner-lehmus

Generalization

- Recently, its generalization, regarding internal as well as *external angle bisectors*, has been approached through automatic tools



Bisectors for internal and external angles at vertex A are constructed intersecting circle of center at A and radius AC with side AB (and its prolongation) at (p, q), then placing lines through A and the midpoints of C and (p, q). The two bisector lines intersect the opposite side CB at (a,b).

The segments from A to (a, b) are the two bisectors at A.

Generalization: formulation

- *If a triangle has two (internal/internal, internal/external, external/internal, external/external) angle-bisectors of different vertices, with the same length, then the triangle is …..????*

- It is a case of automatic discovery.

- Problem: Finding the *missing hypotheses*:

if the triangle is *????*

then at least two (generalized) bisectors for different vertices will have same length.

Why it is challenging

- A booklet on the topic, by the "father" of the algebraic geometry approach to geometry theorem proving:

  W.-t.Wu, X.-l. Lü: "Triangles with equal bisectors". People's Education Press, Beijing, (1985) [in Chinese]

- Recent contributions

  D. Wang (2004), F. Botana (2007)

- Already difficult proof for the standard statement

- Impossibility to deal (in Wang's approach) with the case of *three bisectors*, because it involves two thesis:

  bisectors at A = bisectors at B = bisectors at C

Approach

- We fix A(0,0), B(1,0).

C(x,y) is free.

We look for the locus of C such that some pair of bisectors (at A and B, at A and C, at B and C) have equal length.

- Method: G. Dalzotto, T. Recio: *On protocols for the automated discovery of theorems in elementary geometry*. J. Automated Reasoning, (2009) no. 43, pp. 203-236.

Approach

• Algebraic ideal manipulations: elimination, saturation of ideals.

• Finds equations that should verify (x, y) (i.e. Vertex C) for the statement to hold. They are both necessary and sufficient conditions.

• A few seconds with standard PolynomialIdeal software (Maple package, involving Groebner basis computations).

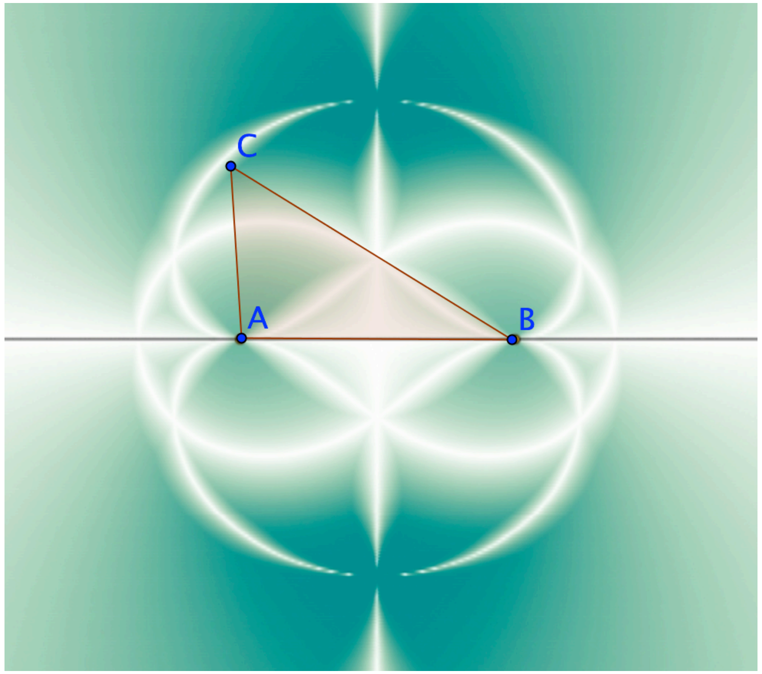
• Method valid for multiple thesis. For instance, for the equality of three bisectors (new!)

Statement for equal bisectors at A and B

If vertex C is placed on the degree ten curve below:

$$14x^2y^4+y^2+246y^2x^6+76x^8-y^6+8x^{10}+9y^{10}-164y^2x^5+12y^4x-10x^2y^2-4x^4-44y^8x-$$
$$136y^4x^3+278y^4x^4-64x^7-164x^7y^2+122y^6x^2-6y^4+8x^5-36y^6x+20y^2x^3+$$
$$84y^4x^6+86x^4y^6+44x^2y^8+16x^6+41y^2x^8+31y^2x^4-40x^9-252y^4x^5-172y^6x^3+14y^8=0$$

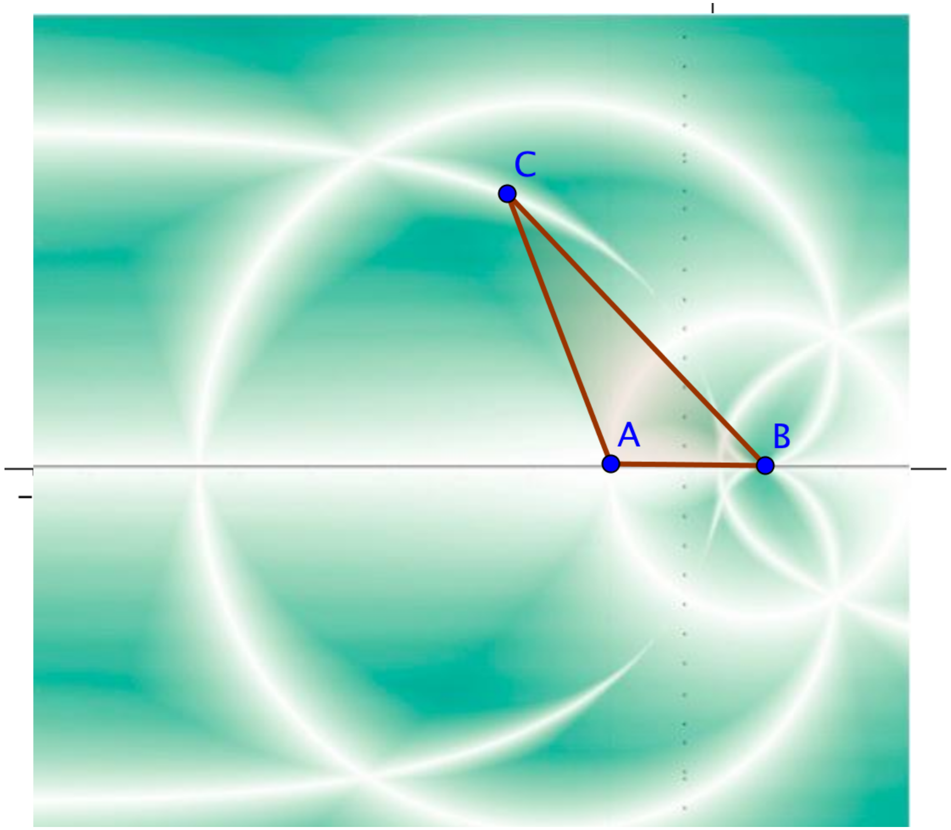
(or at the Y-axis, isosceles case)



Then at least two (generalized) bisectors for vertices A and B will have same length.

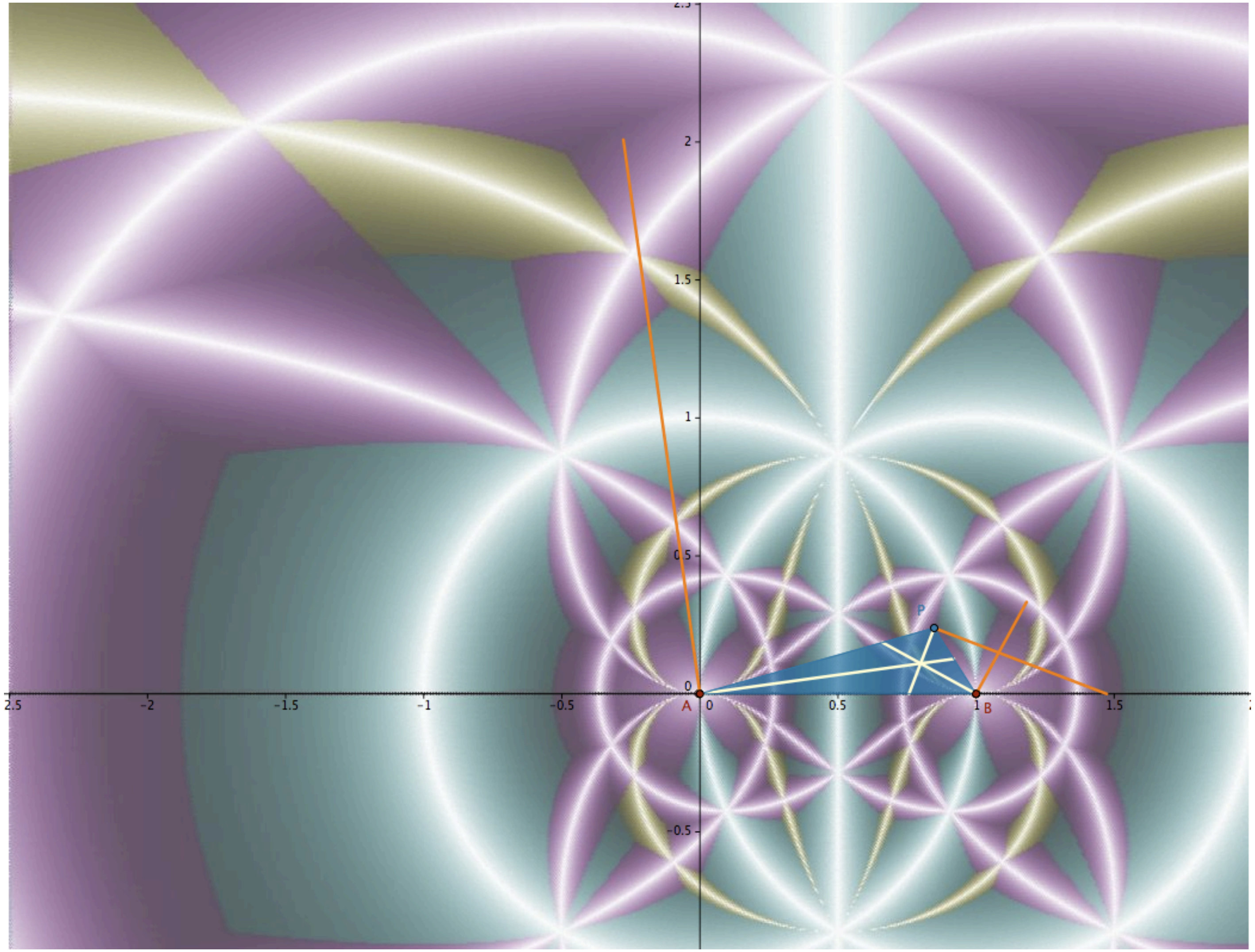
Equality of bisectors of A and C

Vertex C placed at a degree ten curve plus a circle centered at B, radius AB (isosceles case)



Likewise, for B and C (image of curve, simmetry axis x=1/2, dot line)

The degree 35 curve, locus of C for the triangle to have two equal bisectors at some pair of vertices



Three bisectors

The triangle ABC has three equal length generalized bisectors (one at each of the three vertices) if and only if

a) The triangle is equilateral.

b) $x = 2/17 - (2/17)RootOf(4Z^4 + 349Z^2 - 64)^2, y = RootOf(4Z^4 + 349Z^2 - 64)$ (aprox. $x = 0.09611796796, y = +-0.4277818044$). These two points correspond to the equality of lengths for the external bisectors of A and C and the internal bisector of B.

c) Likewise, we have the two points $x = 15/17 + (2/17)RootOf(4Z^4 + 349Z^2 - 64)^2, y = RootOf(4Z^4 + 349Z^2 - 64)$ (aprox. $x = 0.09038820320, y = +-0.4277818044$). These points correspond to the equality of lengths for the external bisectors of B and C and the internal bisector of A.

d) $x = 1/2, y = RootOf(4Z^4 - 19Z^2 - 4)$ (aprox. $x = .5000000000, y = 2.225295714$). These two points correspond to the equality of lengths for the external bisectors of A and B and the internal bisector of C.

Three bisectors

- Remark that there are **no** triangles where two internal bisectors and one external bisector (for different vertices) have equal length, and that there are **no** triangles where the three external bisectors (one for each of the three vertices) are equal (except for the case of innite length).

- Further details:

R. Losada, T. Recio, J. Valcarce: "Sobre el descubrimiento automático de diversas generalizaciones del Teorema de Steiner-Lehmus", Boletín de la Sociedad Puig Adam, no. 82, (2009), pp. 53-76 .